Pricing Defaultable Participating Contracts with Regime Switching and Jump Risk

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Outline

Participating Life Insurance Contract

Participating life insurance contract is the contract that combines the functions of life insurance and financial investment. Policyholders can enjoy the benefits of financial investment in conjunction with mortality protection.

Jump and Regime Switching

The evolution of asset prices includes important features such as jumps and regime switching.

- In the short term, the evolution of asset prices exhibits fairly extreme movements.
- In the long term, structural changes in the macroeconomic conditions or in the business cycles cause modifications in the evolution pattern of asset prices.

References

- Lévy models: Ballotta (IME, 2005), Riesner (IME, 2006), Kassberger, Kiesel, and Liebmann (IME, 2008), Le Courtois and Quittard-Pinon (GRIR, 2008) and Bauer, Bergmann, and Kiesel (AB, 2010).
- Regime switching Brownian motion models: Hardy (NAAJ, 2001), Siu (IME, 2005) and Lin, Tan, and Yang (NAAJ, 2009).
- Regime switching jump diffusion model: Fard and Siu (IME, 2013).

Our Main Contributions

We provide:

- a participating life insurance contract pricing model that combines double exponential jumps and regime switching features in stochastic interest rate environment.
- a numerical method to compute the matrix Wiener-Hopf factorization for the first passage time result of the process based on the results of Jiang and Pistorius (FS, 2008).
- closed-form formulas for the price of participating life insurance contract up to Laplace transform.
- A simple improvement of Cai's one dimensional and two dimensional two-sided Laplace inversion algorithm and define "worst regime" to solve error control problem in regime switching case.
- an empirical study on the influence of regime switching risk, jump risk and credit risk on the price of participating life insurance contract.

Participating Life Insurance Contracts

The insurer is assumed to manage the investment of funds in a specified reference portfolio and the funds are partly financed with the premium L_0. The insured can enjoy benefit of excess investment return from $\alpha = \frac{L_0}{A_0}$ shares of the funds, where A_0 is the initial value of the asset portfolio.

The insured is guaranteed with a minimum rate r_g and hence the minimum guarantee is $L_T = L_0 e^{r_g T}$. Provided the investment of the funds perform well, the insured receives an extra bonus from excess investment return $\delta(\alpha A_T - L_T)$ where

- T is the maturity of life insurance contract.
- A_T is the value of reference portfolio at time T.
- δ is the participating rate.

The Payoff Structure

the bankruptcy time τ of insurer is assumed to happen when the value of reference portfolio falls below a proportion $\kappa < 1$ of the minimum guarantee. Then, the bankruptcy time τ of insurer can be written as

$$\tau = \inf\{t \geqslant 0 : A_t \leqslant \kappa L_0 e^{r_g t}\}.$$

The payoff of the life insurance contract without early default is as follows:

$$\Theta_L(T) = \begin{cases} A_T & \text{if } A_T < L_T \\ L_T & \text{if } L_T \leqslant A_T \leqslant \dfrac{L_T}{\alpha} \\ L_T + \delta(\alpha A_T - L_T) & \text{if } A_T > \dfrac{L_T}{\alpha} \end{cases}$$

The Payoff Structure

The payoff can be written in a simple form as

$$\Theta_L(T) = L_T + \delta(\alpha A_T - L_T)^+ - (L_T - A_T)^+.$$

where the minimum guarantee plus a bonus option concerning the sharing of profits from the investment of funds and a put option related to the default risk of the insurer at maturity.

Allowing the default to happen at any time, the pricing formula under the risk-neutral measure Q is as follows:

$$V = E_Q\left(e^{-\int_0^T r_s \mathrm{d}s}(L_T + \delta(\alpha A_T - L_T)^+ - (L_T - A_T)^+)\mathbb{1}_{\tau \geqslant T} + e^{-\int_0^\tau r_s \mathrm{d}s} A_\tau \mathbb{1}_{\tau < T} \right).$$

The Regime Switching Jump Diffusion Model

The value of the reference portfolio is assumed to follow an exponential regime switching jump diffusion process under the real-world probability measure P:

$$A_t = A_0 e^{X_t},$$

where A_0 is the initial value and X is a regime switching jump diffusion process:

$$X_t = \int_0^t \langle \hat{\mu}, J_s \rangle \mathrm{d}s + \int_0^t \langle \hat{\sigma}, J_s \rangle \mathrm{d}W_s + \int_0^t \mathrm{d}\langle \hat{N}, J_s \rangle$$

where

- J is a continuous time Markov chain process.
- $\hat{\mu}$ and $\hat{\sigma}$ are constant vectors.
- For each state i, $\hat{N}_i$ is a compound Poisson process with rate $\hat{\lambda}_i$ and the jump size is modeled with an asymmetric double exponential distribution.
- W is an independent standard Brownian motion.

The Zero-coupon Bond

The dynamics of the zero-coupon bond value $P(t, T)$ is assumed to follows as

$$P(t, T) = P(0, T)e^{Y_t}$$

where $P(0, T)$ is the initial value and Y is a regime switching Brownian motion:

$$Y_t = \int_0^t \left(r_s - \frac{\sigma_P(s, T)^2}{2} \right) \mathrm{d}s - \int_0^t \langle \hat{\sigma}_P, J_s \rangle \mathrm{d}W_s^1$$

where

- J is a continuous time Markov chain process.
- W^1 is a standard Brownian motion having a correlation coefficient ρ with W.
- For each state i, $\hat{\sigma}_{P,i}$ is an exponential volatility structure.

The First Passage Time Problem

The following proposition is a practical consequence of the Theorem 3 in Jiang and Pistorius (FS, 2008). The proposition is used to deduce closed formulas for the price of life insurance contract and default probability up to Laplace transform.

Lemma 1 Denote τ the first passage time of Z across a constant level b as

$$\tau = \inf\{t > 0 : Z_t > b\}.$$

Denote $\hat{u} = (u, ..., u)$. For any $u > 0$ and $w < \max\{\eta_{11}, ..., \eta_{1n}\}$,

$$E\left(e^{-u\tau + wZ_\tau}\right) = Y_0' W^{(\hat{u},+)} e^{Q^{(\hat{u},+)}(b-z)} e^{wb} \tilde{h}(w)$$

where z is the initial point of Z, Y_0 is the initial state of Y,

$$\tilde{h}(w) = \left(\left(\frac{\eta_{11}}{\eta_{11} - w}\hat{h}_1, ..., \frac{\eta_{1n}}{\eta_{1n} - w}\hat{h}_n\right), \left(\hat{h}_1, ..., \hat{h}_n\right)\right)',$$

and $Q^{(\hat{u},+)}$ is the Wiener-Hopf factor.

- $$W^{(\hat{a},+)} = \begin{pmatrix} I_{2n} \\ \zeta^{(\hat{a},+)} \end{pmatrix}$$
- $\zeta^{(\hat{a},+)}$ and $Q^{(\hat{a},+)}$ are the Wiener-Hopf factors.
- Y_0 is the initial state of Y, which is a continuous time Markov chain process with a finite state space

$$E = \underbrace{E^+}_{\text{Positive jump}} \cup \underbrace{E^0}_{\text{Pure diffusion}} \cup \underbrace{E^-}_{\text{Negative jump}}.$$

The First Passage Time Problem

Let B be the fluid embedding of X. It is a continuous process whose paths are constructed from the paths of X by replacing positive jumps by linear segments with slope $+1$ and negative jumps by linear segments with slope -1.

The up-crossing and down-crossing ladder processes $\tilde{Y}^{+}$, $\tilde{Y}^{-}$ of (B, Y) are defined as time changes of Y that are constructed such that Y is observed only when new maxima and minima of B occur respectively. They are Markov processes. $Q^{(\hat{a},+)}$ and $Q^{(\hat{a},-)}$ are the generator matrices of $\tilde{Y}^{+}$ and $\tilde{Y}^{-}$ and $\zeta^{(\hat{a},+)}$ and $\zeta^{(\hat{a},-)}$ are the corresponding initial distributions.

The quadruple $(\zeta^{(\hat{a},+)}, Q^{(\hat{a},+)}, \zeta^{(\hat{a},-)}, Q^{(\hat{a},-)})$ is a Wiener-Hopf factorization of (B, Y) and the Wiener-Hopf factorization of (B, Y) is unique in the case that $\hat{a} > 0$.

The First Passage Time Problem

Definition 1 Denote as $(G^{(\hat{a},+)}, G^{(\hat{a},-)})$ a pair of irreducible $2n * 2n$ matrices, i.e. matrices with non-negative off-diagonal elements and non-positive row sums and $(\Pi^{(\hat{a},+)}, \Pi^{(\hat{a},-)})$ a pair of $n * 2n$ matrices with sub-probability vectors as rows. The quadruple

$$(\Pi^{(\hat{a},+)}, G^{(\hat{a},+)}, \Pi^{(\hat{a},-)}, G^{(\hat{a},-)})$$

is the Wiener-Hopf factorization of (A, Y) associated with $\hat{a} > 0$ if

$$\Xi(-G^{(\hat{a},+)}, W^{(\hat{a},+)}) = \Xi(G^{(\hat{a},-)}, W^{(\hat{a},-)}) = O,$$

where

$$\Xi(S, W) = \frac{1}{2}\Sigma^2 WS^2 + VWS + Q_{\hat{a}}W,$$

with the $3n \times 3n$ diagonal matrices:

$$\Sigma = \begin{pmatrix} O_n & & \\ & \begin{pmatrix} \hat{\sigma}_1 & & \\ & \ddots & \\ & & \hat{\sigma}_n \end{pmatrix} & \\ & & O_n \end{pmatrix}, \quad V = \begin{pmatrix} I_n & & \\ & \begin{pmatrix} \hat{\mu}_1 & & \\ & \ddots & \\ & & \hat{\mu}_n \end{pmatrix} & \\ & & -I_n \end{pmatrix},$$

and the $3n \times 2n$ matrices:

$$W^{(\hat{a},+)} = \begin{pmatrix} I_{2n} \\ \Pi^{(\hat{a},+)} \end{pmatrix} \quad W^{(\hat{a},-)} = \begin{pmatrix} \Pi^{(\hat{a},-)} \\ I_{2n} \end{pmatrix}$$

where I_n and I_{2n} are identity matrices of size $n \times n$ and of size $2n \times 2n$, respectively.

Numerical Method

The following numerical algorithm can be used to compute the related Wiener-Hopf factors $\zeta^{(\hat{a},+)}$ and $Q^{(\hat{a},+)}$.

Numerical Algorithm: Computation of the $\zeta^{(\hat{a},+)}$ and $Q^{(\hat{a},+)}$

- Step 1: Compute $2n$ roots $Re(\beta_1) \leqslant Re(\beta_2) \leqslant ... \leqslant Re(\beta_{2n}) < 0$ from the equation $f(\beta) = 0$ where $f(\beta) = \det(K(\beta)) = 0$ and $K(\beta) = \frac{1}{2}\Sigma^2\beta^2 - V\beta + Q_{\hat{a}}$.
- Step 2: For $i = 1, 2, ..., 2n$, compute $3n$ vector γ_i by solving a system of linear equations $K(\beta_i)\gamma_i = 0$.
- Step 3: Set $\vartheta_i = (\gamma_{i,1}, ..., \gamma_{i,2n})'$, $i = 1, 2, .., 2n$ and $Z = [\vartheta_1, \vartheta_2, ..., \vartheta_{2n}]$, then
$$Q^{(\hat{a},+)} = Z\,\mathrm{diag}\{\beta_1, \beta_2, ..., \beta_{2n}\}\,Z^{-1},$$
and the matrix exponential is computed as
$$e^{Q^{(\hat{a},+)}x} = Z\,\mathrm{diag}\{e^{\beta_1 x}, e^{\beta_2 x}, ..., e^{\beta_{2n} x}\}\,Z^{-1}.$$
- Step 4: For $k = 2n+1, 2n+2, ..., 3n$, compute $2n$ vector ξ_k by solving a system of linear equations $Z'\xi_k = (\gamma_{1,k}, ..., \gamma_{2n,k})'$. Then
$$\zeta^{(\hat{a},+)} = [\xi_1, \xi_2, ..., \xi_n]'.$$

The Price of Participating Life Insurance Contract

Define the forward-neutral measure Q_T by

$$\frac{\mathrm{d}Q^T}{\mathrm{d}Q} = e^{-\int_0^T r_s \mathrm{d}s} \frac{P(T,T)}{P(0,T)} = e^{-\int_0^T \sigma_P(s,T)\mathrm{d}W_s^1 - \frac{1}{2}\int_0^T \sigma_P(s,T)^2 \mathrm{d}s}.$$

We have the following proposition:

Proposition 1 The process $\{X_t; t \geqslant 0\}$ keeps a regime switching double exponential jump diffusion structure under Q^T. Let X be defined under Q^T as follows:

$$X_t = \int_0^t \mu_s^T \mathrm{d}s + \int_0^t \sigma_s^T \mathrm{d}W_s^T + \int_0^t \mathrm{d}N_s^T.$$

Then, W^T defined by $\mathrm{d}W_t^T = \rho \mathrm{d}W_t^{T,1} + \sqrt{1-\rho^2}\mathrm{d}W_t^{T,2}$ is a standard Brownian motion where $W^{T,1}$ and $W^{T,2}$ are two independent Brownian motions under Q^T such that $< \mathrm{d}W_t^{T,1}, \mathrm{d}W_t^{T,2} >= 0$ and $W_t^{T,1} = W_t^1 + \int_0^t \sigma_P(s,T)\mathrm{d}s$ and $W^{T,2} = W^2$. The $\hat{\mu}_i^T = \hat{\mu}_i - \rho\hat{\sigma}_i\hat{\sigma}_{P,i}$, $\hat{\sigma}_i^T = \hat{\sigma}_i$ and $N^T = N$.

The Price of Participating Life Insurance Contract

Define another probability measure Q^* as $\frac{\mathrm{d}Q^*}{\mathrm{d}Q} = e^{-\int_0^T r_s \mathrm{d}s} \frac{A_T}{A_0}$. We have the following propositions:

Proposition 2 The process $\{X_t; t \geqslant 0\}$ keeps a regime switching double exponential jump diffusion structure under Q^*. Let X be defined under Q^* as follows:

$$X_t = \int_0^t \mu_s^* \mathrm{d}s + \int_0^t \sigma_s^* \mathrm{d}W_s^* + \int_0^t \mathrm{d}N_s^*.$$

Then, W^* defined by $W_t^* = W_t - \int_0^t \sigma_s \mathrm{d}s$ is a standard Brownian motion, $\hat{\sigma}_i^* = \hat{\sigma}_i$, $\hat{\lambda}_i^* = \hat{\lambda}_i \omega_i$,
$p_i^* = \frac{1}{\omega_i}\left(\frac{p_i \hat{\eta}_{1i}}{\hat{\eta}_{1i} - 1}\right)$, $\hat{\eta}_{1i}^* = \hat{\eta}_{1i} - 1$ and $\hat{\eta}_{2i}^* = \hat{\eta}_{2i} + 1$, where
$\omega_i = E_P(e^{Y_i}) = \frac{p_i \hat{\eta}_{1i}}{\hat{\eta}_{1i} - 1} + \frac{(1 - p_i)\hat{\eta}_{2i}}{\hat{\eta}_{2i} + 1}$.

Proposition 3 If τ is an $\mathcal{F}$-stopping time then on $\{\tau < \infty\}$,

$$\frac{\mathrm{d}Q^*}{\mathrm{d}Q}|_{\mathcal{F}_\tau} = e^{-\int_0^\tau r_s \mathrm{d}s} \frac{A_\tau}{A_0}.$$

The Price of Participating Life Insurance Contract

The pricing formula is written in a simple form as follows:

$$V = GF + BO - PO + LR \quad ,$$

where

$$\begin{cases} GF = E_Q\left(e^{-\int_0^T r_s \mathrm{d}s} L_T \mathbb{1}_{\tau \geqslant T} \right) \\ BO = E_Q\left(e^{-\int_0^T r_s \mathrm{d}s} \delta(\alpha A_T - L_T)^+ \mathbb{1}_{\tau \geqslant T} \right) \\ PO = E_Q\left(e^{-\int_0^T r_s \mathrm{d}s} (L_T - A_T)^+ \mathbb{1}_{\tau \geqslant T} \right) \\ LR = E_Q\left(e^{-\int_0^\tau r_s \mathrm{d}s} A_\tau \mathbb{1}_{\tau < T} \right) \end{cases}$$

and GF, BO, PO, LR corresponds to the minimum guarantee, the bonus option, the put option related to default risk and the recovery from the default before the maturity, respectively.

The Price of Participating Life Insurance Contract

We denote $Z_t = -X_t + r_g t$ and τ can be rewritten as

$$\tau = \inf\{t \geqslant 0 : Z_t \geqslant \ln \frac{A_0}{\kappa L_0}\}.$$

Then, we have:

- $GF:$ $\quad P(0,T)L_T(1 - Q^T(\tau < T))$
- $BO:$ $$\delta\alpha A_0 \left(Q^* \left(Z_T \leqslant \ln \frac{\alpha A_0}{L_0} \right) - Q^* \left(Z_T \leqslant \ln \frac{\alpha A_0}{L_0}, \tau < T \right) \right) \\ -\delta P(0,T) L_T \left(Q^T \left(Z_T \leqslant \ln \frac{\alpha A_0}{L_0} \right) - Q^T \left(Z_T \leqslant \ln \frac{\alpha A_0}{L_0}, \tau < T \right) \right)$$
- $PO:$ $$P(0,T)L_T \left(1 - Q^T \left(Z_T < \ln \frac{A_0}{L_0} \right) - Q^T (\tau < T) + Q^T \left(Z_T < \ln \frac{A_0}{L_0}, \tau < T \right) \right) \\ -A_0 \left(1 - Q^* \left(Z_T < \ln \frac{A_0}{L_0} \right) - Q^* (\tau < T) + Q^* \left(Z_T < \ln \frac{A_0}{L_0}, \tau < T \right) \right)$$
- $LR:$ $\quad A_0 Q^* (\tau < T)$

The Price of Participating Life Insurance Contract

The regime switching double exponential jump diffusion structure keeps unchanged under the measure Q^T and Q^*. Then, we only need to compute these three items

- $E_1 = Q(\tau < t)$
- $E_2 = Q(Z_t < z)$
- $E_3 = Q(Z_t < z, \tau < t)$.

The Price of Participating Life Insurance Contract

We have Laplace transform of these probabilities as follows:

- $$\begin{aligned}\hat{f}(u) &= \int_0^\infty e^{-ut} Q(\tau < t)\mathrm{d}t \\ &= \frac{1}{u} Y_0' W^{(\hat{u},+)} e^{Q^{(\hat{u},+)}(\ln \frac{A_0}{\kappa L_0} - x)} \mathbf{1}\end{aligned}$$

- $$\begin{aligned}\hat{g}(v) &= \int_{-\infty}^\infty e^{-vz} Q(Z_t < z)\mathrm{d}z \\ &= \frac{1}{v} J_0 e^{G(-v)t} \mathbf{1}.\end{aligned}$$

- $$\begin{aligned}\hat{k}(u,v) &= \int_0^\infty \int_{-\infty}^\infty e^{-ut-vz} Q(Z_t < z, \tau < t)\mathrm{d}z\mathrm{d}t \\ &= \frac{1}{v} Y_0' W^{(\hat{u},+)} e^{Q^{(\hat{u},+)}(\ln \frac{A_0}{\kappa L_0} - x)} (\frac{\kappa L_0}{A_0})^v \tilde{H}(-v)(uI - G(-v))^{-1} \mathbf{1}\end{aligned}$$

The Price of Participating Life Insurance Contract

where

- Q is the generator matrix of J
- $\varphi_j(u) = \hat{\mu}_j u + \frac{1}{2}\hat{\sigma}_j^2 u^2 + \hat{\lambda}_j\left(\frac{p_j\hat{\eta}_{1j}}{\hat{\eta}_{1j} - u} + \frac{q_j\hat{\eta}_{2j}}{\hat{\eta}_{2j} + u} - 1\right)$
- $G(u) = Q + \text{diag}\{\varphi_j(u)\}$
-

$$\tilde{H}(w) = \begin{pmatrix} \begin{pmatrix} \frac{\hat{\eta}_{11}}{\hat{\eta}_{11} - w} & & \\ & \ddots & \\ & & \frac{\hat{\eta}_{1n}}{\hat{\eta}_{1n} - w} \end{pmatrix} \\ I_n \end{pmatrix}.$$

Numerical Method

- To compute $Q(\tau < t)$, we use the Abate¨CWhitt algorithm to perform the numerical one-sided Laplace inversion.
- To compute $Q(Z_t < z)$, we use the numerical two-sided Laplace inversion algorithm in Cai, Kou, and Liu (2014).
- To compute $Q(Z_t < z, \tau < t)$, we use the two-dimensional two-sided Laplace inversion in Cai and Shi (2015).

Numerical Method

The inversion formula in Cai, Kou, and Liu (2014) to get $f(t)$ from its two-sided Laplace transform $L_f(s)$ is as follows:

$$f(t;\sigma,C) = -\frac{e^{\sigma t}L_f(\sigma)}{2(|t|+C)} + \frac{e^{\sigma t}}{|t|+C}\sum_{k=0}^{\infty}\left[(-1)^k \Re\left(e^{-\frac{\mathrm{sgn}(t)Ck\pi i}{t+\mathrm{sgn}(t)C}} \times L_f\left(\sigma + \frac{k\pi i}{t+\mathrm{sgn}(t)C}\right)\right)\right].$$

The inversion formula in Cai and Shi (2015) to obtain $f(t_1, t_2)$ from its Laplace transform $L_f(u_1, u_2)$ is as follows:

$$\begin{aligned} f(t_1, t_2; v_1, v_2, C_1, C_2) &= \frac{e^{v_1 t_1 + v_2 t_2}}{4(|t_1| + C_1)(|t_2| + C_2)} \times \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}(-1)^{k_1+k_2} \\ &\Re\left(e^{-\sum_{j=1}^{2}\frac{k_j C_j \mathrm{sgn}(t_j)\pi i}{t_j + C_j \mathrm{sgn}(t_j)}} L_f\left(v_1 + \frac{k_1\pi i}{t_1 + C_1\mathrm{sgn}(t_1)}, v_2 + \frac{k_2 \pi i}{t_2 + C_2 \mathrm{sgn}(t_2)}\right)\right). \end{aligned}$$

Numerical Method

As in Choudhury, Lucantoni, and Whitt (1994), in order to use the Euler summation technique for alternating series, we use the $f(\overline{t_1}, \overline{t_2}) = \overline{f(t_1, t_2)}$ and obtain the inversion formula as follows:

$$
\begin{aligned}
f(t_1, t_2; v_1, v_2, C_1, C_2) = & \frac{e^{v_1 t_1 + v_2 t_2}}{4(|t_1| + C_1)(|t_2| + C_2)} \Bigg(L_f(v_1, v_2) \\
& -2 \sum_{k_1=0}^{\infty} (-1)^{k_1} \Re \left(e^{-\frac{k_1 C_1 \operatorname{sgn}(t_1) \pi i}{t_1 + C_1 \operatorname{sgn}(t_1)}} L_f\left(v_1 + \frac{k_1 \pi i}{t_1 + C_1 \operatorname{sgn}(t_1)}, v_2\right)\right) \\
& -2 \sum_{k_2=0}^{\infty} (-1)^{k_2} \Re \left(e^{-\frac{k_2 C_2 \operatorname{sgn}(t_2) \pi i}{t_2 + C_2 \operatorname{sgn}(t_2)}} L_f\left(v_1, v_2 + \frac{k_2 \pi i}{t_2 + C_2 \operatorname{sgn}(t_2)}\right)\right) \\
& +2 \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} (-1)^{k_1} \\
& \Re \left((-1)^{k_2} e^{-\sum_{j=1}^{2} \frac{k_j C_j \operatorname{sgn}(t_j) \pi i}{t_j + C_j \operatorname{sgn}(t_j)}} L_f\left(v_1 + \frac{k_1 \pi i}{t_1 + C_1 \operatorname{sgn}(t_1)}, v_2 + \frac{k_2 \pi i}{t_2 + C_2 \operatorname{sgn}(t_2)}\right)\right) \\
& +2 \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} (-1)^{k_1} \Re \left((-1)^{k_2} e^{-\frac{k_1 C_1 \operatorname{sgn}(t_1) \pi i}{t_1 + C_1 \operatorname{sgn}(t_1)} + \frac{k_2 C_2 \operatorname{sgn}(t_2) \pi i}{t_2 + C_2 \operatorname{sgn}(t_2)}} \right. \\
& \left. L_f\left(v_1 + \frac{k_1 \pi i}{t_1 + C_1 \operatorname{sgn}(t_1)}, v_2 - \frac{k_2 \pi i}{t_2 + C_2 \operatorname{sgn}(t_2)}\right)\right)\Bigg)
\end{aligned}
$$

Numerical Method

As in Abate and Whitt (1992) and Choudhury, Lucantoni, and Whitt (1994), we use the Euler transformation to calculate the alternating series of the form $S = \sum_{k=0}^{\infty} (-1)^k a_k$ as it accelerates the convergence rate. The Euler sum with parameters n and m is as follows:

$$E(m,n) = \sum_{k=0}^{m} \binom{m}{k} 2^{-m} S_{n+k}$$

where

$$S_j = \sum_{k=0}^{j} (-1)^k a_k.$$

Error Control for Numerical Laplace Inversion

In the regime switching jump diffusion case, we define a "worst state", in which the process combines the smallest drift, the largest volatility, jump intensity, negative jumps and no positive jumps among all the states. We can express the process in "worst state" as follows:

$$\tilde{Z}_t = \tilde{\mu} t + \tilde{\sigma} W_t + \sum_{i=1}^{N(t)} Y_i$$

where $\tilde{\mu} = \min\{\mu_1, ..., \mu_n\}$, $\tilde{\sigma} = \max\{\sigma_1, ..., \sigma_n\}$, W a standard Brownian motion, N a Poisson process with rate $\tilde{\lambda} = \max\{\lambda_1, ..., \lambda_n\}$, and the jump sizes $\{Y_1, Y_2, ...\}$ independent and identically distributed random variables having an exponential distribution with the density function

$$f_Y(y) = \tilde{\eta}_2 e^{\tilde{\eta}_2 y}$$

where $\tilde{\eta}_2 = \min\{\eta_1, ..., \eta_n\}$. The stochastic processes W, N and stochastic variables $\{Y_1, Y_2, ...\}$ are independent.

Discretization Errors in Computing $Q(Z_t < z)$

Lemma 2 If there exists a nonnegative function $\delta(*)$ such that, for any $\sigma \in [\sigma_l^*, \sigma_u^*]$, we have

$$e^{-\sigma z} Q(Z_t < z) \leqslant \delta(\sigma) < +\infty \quad \text{for any} \quad z,$$

then, for any fixed $z \in \mathbb{R}$, $\sigma \in (\sigma_l^*, \sigma_u^*)$, and $C > 0$, we have the error bound

$$|e_D(z, \sigma, C)| \leqslant \frac{\rho(\sigma, z)}{e^{\theta(\sigma)C} - 1}$$

where $\theta(\sigma) = 2\min\{\sigma_u^* - \sigma, \sigma - \sigma_l^*\} > 0$ and

$$\rho(\sigma, z) = \begin{cases} \delta(\sigma_u^*) e^{(2\sigma - \sigma_u^*)z} + \delta(\sigma_l^*) e^{(3\sigma_l^* - 2\sigma)z} & \text{if } z \geqslant 0, \\ \delta(\sigma_l^*) e^{(2\sigma - \sigma_l^*)z} + \delta(\sigma_u^*) e^{(3\sigma_u^* - 2\sigma)z} & \text{if } z < 0. \end{cases}$$

The upper bound of discretization error can be computed by making $\delta(v) = E(e^{-v\tilde{Z}_t})$.

Truncation Errors in Computing $Q(Z_t < z)$

Lemma 3 For any fixed $t \in \mathbb{R}$, $\sigma \in ROAC$, and $C \geqslant 0$ such that $|t| + C > 0$, the following statements hold.

1. If there exist $\rho > 0$, $\omega^* \geqslant 0$, and $\sigma(\sigma) > 0$ such that

$$|\hat{g}(\sigma + i\omega)| \leqslant \sigma(\sigma)|\omega|^{-(1+\rho)} \quad \text{for all} \quad |\omega| > \omega^*,$$

then the truncation error

$$|e_T(t, \sigma, C, N)| \leqslant \frac{\sigma(\sigma)e^{\sigma t}(|t| + C)^{\rho}}{\rho\, \pi^{1+\rho}} N^{-\rho} = O(N^{-\rho})$$

for any $N \in \mathbb{N}$ such that $N > (|t| + C)\,\omega^*/\pi - 1$.

2. If there exit $\beta \in \mathbb{R}$, $\xi > 0$, $\rho > 0$, $\omega^* \geqslant 0$, and $\sigma(\sigma) > 0$ such that

$$|\hat{g}(\sigma + i\omega)| \leqslant \sigma(\sigma)|\omega|^{-\beta} e^{-\rho|\omega|^{\xi}} \quad \text{for all} \quad |\omega| > \omega^*,$$

then the truncation error

$$|e_T(t, \sigma, C, N)| \leqslant \frac{\sigma(\sigma)e^{\sigma t}}{\pi\, \xi\, \rho^{(1-\beta)/\xi}} \gamma\left(\frac{1-\beta}{\xi}, \rho\,\alpha N^{\xi}\right) = O(N^{1-\beta-\xi} e^{-\rho\,\alpha N^{\xi}})$$

for any $N \in \mathbb{N}$ such that $N > (|t| + C)\,\omega^*/\pi - 1$. Here $\alpha = (\pi/(|t| + C))^{\xi} > 0$, and, for any $s \in \mathbb{R}$, $\Gamma(s, x) = \int_x^{\infty} y^{s-1} e^{-y} \mathrm{d}y$ denotes the upper incomplete gamma function.

The upper bound of truncation error can be computed by specifying parameters in Lemma 3 as follows:

$$\sigma(\sigma) = e^{t\left(\frac{\tilde{\sigma}^2}{2}\sigma^2 - \tilde{\mu}\sigma + \lambda\left(\frac{\tilde{\eta}_2}{\tilde{\eta}_2 - \sigma} - 1\right)\right)},$$

$$\beta = 1, \rho = \frac{t\tilde{\sigma}^2}{2}, \xi = 2, \omega^* = 0.$$

Discretization Errors to Compute $Q(Z_t < z, \tau < t)$

Lemma 4 If there exists a function $\delta(*)$ such that, for any $(u, v) \in [l_1^*, u_1^*] \times [l_2^*, u_2^*]$,

$$e^{-ut-vz}|Q(Z_t < z, \tau < t)| \leqslant \delta(u, v) < +\infty \quad \text{for any} \quad (t, z) \in \mathbb{R}^2,$$

then for any $(u, v) \in [l_1^*, u_1^*] \times [l_2^*, u_2^*]$, $(C_1, C_2) \in \mathbb{R}_+^2$ and $(t, z) \in \mathbb{R}^2$, the discretization error has the following bound

$$|e_D(t, z, u, v, C_1, C_2)| \leqslant \frac{\rho(u, v, t, z)}{(e^{2d_1 C_1} - 1)(e^{2d_2 C_2} - 1)} + \frac{\rho_1(u, t)}{e^{2d_1 C_1} - 1} + \frac{\rho_2(v, z)}{e^{2d_2 C_2} - 1},$$

where $d_1 = \min\{u - l_1^*, u_1^* - u\}$, $d_2 = \min\{v - l_2^*, u_2^* - v\}$,

$$\rho_1(u, t) = \delta(l_1^*, l_2^*)e^{l_1^* t - 2(u - l_1^*)|t|} + \delta(u_1^*, l_2^*)e^{u_1^* t - 2(u_1^* - u)|t|},$$

$$\rho_2(v, z) = \delta(l_1^*, l_2^*)e^{l_2^* z - 2(v - l_2^*)|z|} + \delta(l_1^*, u_2^*)e^{u_2^* z - 2(u_2^* - v)|z|},$$

and

$$\begin{aligned} \rho(u, v, t, z) \quad &= \delta(l_1^*, l_2^*)e^{l_1^* t - 2(u - l_1^*)|t| + l_2^* z - 2(v - l_2^*)|z|} \\ &+ \delta(u_1^*, l_2^*)e^{u_1^* t - 2(u_1^* - u)|t| + l_2^* z - 2(v - l_2^*)|z|} \\ &+ \delta(u_1^*, u_2^*)e^{u_1^* t - 2(u_1^* - u)|t| + u_2^* z - 2(u_2^* - v)|z|} \\ &+ \delta(l_1^*, u_2^*)e^{l_1^* t - 2(u - l_1^*)|t| + u_2^* z - 2(u_2^* - v)|z|}. \end{aligned}$$

Moreover, it follows that $\lim_{C_1, C_2 \to \infty} e_D(t, z, u, v, C_1, C_2) = 0$.

The upper bound of discretization error can be computed by making $\delta(u, v) = e^{G(-v)t}$.

Truncation Errors to Compute $Q(Z_t < z, \tau < t)$

Lemma 5 For any fixed $(t, z) \in \mathbb{R}^2$, $(u, v) \in ROAC$, and $(C_1, C_2) \geqslant 0$ such that $|t| + C_1 > 0$, $|z| + C_2 > 0$,

1. If there exist $\alpha_p > 1$ for $p = 1, \ldots, 4$, $(M_1, M_2) \geqslant 0$, and positive functions $\zeta(u, v)$ and $\zeta_j(u, v, \omega_j)$ for $j = 1, 2$ such that

$$|L_f(u+i\omega_1, v+i\omega_2)| \leqslant \begin{cases} \zeta_2(u, v, \omega_2)|\omega_1|^{-\alpha_1}, & \text{for all } (\omega_1, \omega_2) \in \{(\omega_1, \omega_2) : |\omega_1| > M_1, |\omega_2| \leqslant M_2\}; \\ \zeta_1(u, v, \omega_1)|\omega_2|^{-\alpha_2}, & \text{for all } (\omega_1, \omega_2) \in \{(\omega_1, \omega_2) : |\omega_1| \leqslant M_1, |\omega_2| > M_2\}; \\ \zeta(u, v)|\omega_1|^{-\alpha_3}|\omega_2|^{-\alpha_4}, & \text{for all } (\omega_1, \omega_2) \in \{(\omega_1, \omega_2) : |\omega_1| > M_1, |\omega_2| > M_2\}. \end{cases}$$

then the truncation error

$$|e_T(t, z, u, v, C_1, C_2, N_1, N_2)| \leqslant \frac{e^{u*t+v*z}}{2(|t| + C_1)(|z| + C_2)}[B_1(N_1, N_2) + B_2(N_1, N_2) + B_3(N_1, N_2)]$$

for any $(N_1, N_2) \in \mathbb{N}^2$ such that $N_j > \frac{M_j}{b_j} - 1$ with $b_1 = |a_1| = \frac{\pi}{|t| + C_1} > 0$ and $b_2 = |a_2| = \frac{\pi}{|z| + C_2} > 0$, where

$$\begin{aligned} B_1(N_1, N_2) &= \frac{b_1^{-\alpha_1}}{\alpha_1 - 1} N_1^{1-\alpha_1} \sum_{k_2=-N_2}^{N_2} \zeta_2(u, v, k_2 b_2 \operatorname{sgn}(t_2)) \\ &= O(N_1^{1-\alpha_1}), \quad \text{for any fixed} N_2, \end{aligned}$$

$$\begin{aligned} B_2(N_1, N_2) &= \frac{b_2^{-\alpha_2}}{\alpha_2 - 1} N_2^{1-\alpha_2} \sum_{k_1=-N_1}^{N_1} \zeta_1(u, v, k_1 b_1 \operatorname{sgn}(t_1)) \\ &= O(N_2^{1-\alpha_2}), \quad \text{for any fixed} N_1, \end{aligned}$$

$$\begin{aligned} B_3(N_1, N_2) &= 2\zeta(u, v) \frac{b_1^{-\alpha_3}}{\alpha_3 - 1} \frac{b_2^{-\alpha_4}}{\alpha_4 - 1} N_1^{1-\alpha_3} N_2^{1-\alpha_4} \\ &= O(N_1^{1-\alpha_3} N_2^{1-\alpha_4}); \end{aligned}$$

2. ...

Truncation Errors to Compute $Q(Z_t < z, \tau < t)$

The upper bound of truncation error can be computed by specifying parameters in Lemma 5 as follows:

$$\alpha_1 = 2,\ \alpha_2 = 3,\ \alpha_3 = \frac{3}{2},\ \alpha_4 = 2,$$

$$M_j = \max\{M_j^*, M_j^+\}$$

where

$$M_1^* = \left(\frac{5\tilde{\lambda}}{\tilde{\sigma} M_2^*}\right)^2 \left(\frac{\tilde{\eta}_2}{M_2^*} + 1\right)^2,$$

$$M_2^* = \max\left\{\sqrt{\frac{4}{\tilde{\sigma}^2}(\lambda + |c|) + \left(\frac{4}{\tilde{\sigma}^2}\lambda \tilde{\eta}_2\right)^{\frac{2}{3}}},\ \frac{2\sqrt{2}}{\sigma^2} \times \sqrt{(\tilde{\sigma}^2 v + \tilde{\mu})^2 + |c|\,\tilde{\sigma}^2}\right\},$$

$c = u - \frac{1}{2}\tilde{\sigma}^2 v^2 - \mu v$ and

$$M_1^+ = \max\{Y_1, Y_2, Y_3, Y_4\},$$

$$\begin{cases} Y_1 = 2\left(\sigma \tilde{\eta}_2 + \frac{2|\tilde{\mu}|}{\tilde{\sigma}}\right)^2 \\ Y_2 = \frac{1}{2}\left(\frac{4|\tilde{\mu}|}{\tilde{\sigma}} + \frac{\tilde{\sigma}}{|\tilde{\mu}|}(u + 2\tilde{\lambda})\right)^2 \\ Y_3 = 2\tilde{\eta}_2^2 \tilde{\sigma}^2 + 2\tilde{\eta}_2 |\mu| + \lambda + u + \tilde{\lambda}\tilde{\eta}_2 \\ Y_4 = \frac{Y_3}{\tilde{\eta}_2} \end{cases}$$

$$M_2^+ = 1.$$

$$\zeta_2(u, v, \omega_2) = (\hat{\xi}(u, v, \omega_2))^{-1} \xi(u, v, \omega_2),$$

$$\zeta_1(u, v, \omega_1) = 4\tilde{\sigma}^{-2} \xi_1(u, v, \omega_1)$$

$$\zeta(u, v) = 2\tilde{\sigma}^{-1} \xi(u, v)$$

Truncation Errors to Compute $Q(Z_t < z, \tau < t)$

where

$$\hat{\xi}(u, v, \omega_2) = \begin{cases} \dfrac{(u - \Re(G(-v + \omega_2 i)))}{|u - G(-v + \omega_2 i)|}, & \text{for } u \neq G(-v + \omega_2 i) \\ 1, & \text{for } u = G(-v + \omega_2 i). \end{cases}$$

$$\begin{cases} \xi_1(u, v, \omega_1) = e^{bv} \left(\left| \dfrac{e^{-b\beta_1}(\eta_2 + \beta_1) + e^{-b\beta_2}(-\beta_2 - \eta_2)}{\beta_2 - \beta_1} \right| + \dfrac{|(e^{-b\beta_1} - e^{-b\beta_2})(-\beta_2 - \eta_2)(\eta_2 + \beta_1)|}{|\beta_2 - \beta_1|(\eta_2 + v)} \right) \\ \xi_2(u, v, \omega_2) = \dfrac{e^{bv}}{T_0}(T_1 + T_2 + T_3) \\ \xi(u, v) = \xi_2(u, v, 0) \end{cases}$$

Truncation Errors to Compute $Q(Z_t < z, \tau < t)$

where

$$\begin{cases} T_0 = |z_{M_1} - \eta_2| - \frac{Y_3}{M_1} - \frac{2|\mu|}{\sigma^2} \\ T_1 = e^{b(\eta_2 - \frac{Y_3}{M_1})} Y_3 \\ T_2 = \frac{16\sigma^2}{b^2} e^{\frac{b}{2}(\frac{\sqrt{|\omega_1|}}{\sigma} - \frac{2|\mu|}{\sigma^2})} (|z_{M_1} - \eta_2| - \frac{2|\mu|}{\sigma^2}) \\ T_3 = \frac{Y_3}{|\eta_2 + v + \omega_2 i|} (e^{b(\eta_2 - \frac{Y_3}{M_1})} + e^{b(\frac{\sqrt{M_1}}{\sigma} - \frac{2|\mu|}{\sigma^2})}) (|z_{M_1} - \eta_2| - \frac{2|\mu|}{\sigma^2}) \end{cases}$$

where $z_{M_1} = \frac{\sqrt{M_1}}{\sigma}(1 + i\mathrm{sgn}(\omega_1))$ and β_1, β_2 are two roots with positive reat parts from the equation $G(-z) = u + \omega_1 i$ that satisfy

$$|\beta_1 - \eta_2| \leqslant \frac{Y_3}{|y|}$$

and

$$|\beta_2 - z_{\omega_1}| \leqslant \frac{2|\tilde{\mu}|}{\tilde{\sigma}^2}$$

when $|y| > Y(u)$ and where $z_y = \frac{\sqrt{|y|}}{\sigma}(1 + i \cdot \mathrm{sgn}(y))$.

Illustration

With weights of financial investment in the reference portfolio, we compute the market value of reference portfolio through weighted sum of market value of stocks and bonds, which are indicated by corresponding daily market indices between January 3, 2000 and February 10, 2017.

Table: The financial investment in the assets portfolio

	Proportion of Assets
Assets	
Stocks	20%
Corporate Bonds	40%
Government Bonds	40%

Table: The market indices for the stocks and bonds

	Market Indices
Stocks	The Bloomberg European 500 Index
Corporate Bonds	The BofA Merrill Lynch AAA Euro Broad Market Index
	The BofA Merrill Lynch AA Euro Broad Market Index
	The BofA Merrill Lynch A Euro Broad Market Index
	The BofA Merrill Lynch BBB Euro Broad Market Index
Government Bonds	The BofA Merrill Lynch All Maturity All Euro Government Index

Illustration

We estimate the parameters of the models JD, RSBM and RSJD using the market value of reference portfolio by performing a likelihood maximization.

Table: Parameter estimation for the JD, RSBM and RSJD

Model	Parameters													
	q_{12}	q_{21}	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	p_1	p_2	$\hat{\eta}_{11}$	$\hat{\eta}_{21}$	$\hat{\eta}_{12}$	$\hat{\eta}_{22}$
JD			0.9448		0.0144		86.9533		0.6630		34.8500		11.5942	
RSBM	0.0918	0.0247	-0.0273	0.0790	1.1369	0.5283								
RSJD	0.0668	0.0249	-0.1755	-0.1606	1.0121	0.4851	122.5445	16.1505	0.5113	0.3787	133.2804	21.8127	191.0421	256.6109

- The double exponential jump diffusion model (JD), the regime switching Brownian motion model (RSBM) and the regime switching double exponential jump diffusion model (RSJD).
- For the RSBM and RSJD, the sample likelihood is computed by using the Hamilton filter method in Hamilton (1989).
- For the JD and RSJD, the density of the distribution is not available in analytical form and the probability densities are computed by performing a Fourier inversion of the corresponding characteristic function.
- For simple statement, we only consider two regimes for RSBM and RSJD and the general n regimes case can be easily computed by changing 2×2 matrix into $n \times n$ matrix in computation.

Illustration

Table: Other parameters

r	0.0205
r_g	0.015
A_0	100
α	0.75
δ	0.9
κ	0.75
T	10
J_0	(1,0)'
Y_0	(1/3,0,1/3,0,1/3,0)'

- We use the BofA Merrill Lynch Euro Treasury Bill Index to compute the r.
- Some other parameters setting refer to Le Courtois and Quittard-Pinon (GRIR, 2008).

Illustration

The result shows the RSJD achieves the best fit. The comparison among these models indicates that there exists significant phenomenon of regime switching and the characterization of regime switching risk is more important than that of jump risk.

Table: Loglikelihoods, AIC and BIC for the JD, RSBM and RSJD

	JD	RSBM	RSJD
LogLik	-6808.3	-4339.6	-4320.9
AIC	13629	8691.2	8669.9
BIC	13667	8729.5	8759.2

Illustration

The Figure shows the characterization of regime switching risk increases a lot in the default probabilities. From an intuitive understanding, the high risk and low risk are compromised when the regime switching is not characterized, which results in lowering the default probabilities.

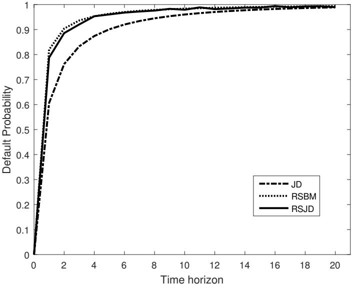

Figure: Default Probability

Thank you for your attention.