Intensity of Preferences in the Presence of Bivariate Risks

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Outline of the Talk

- Selected References
- Previous Results
- A Preliminary Lottery
- The Case of Lotteries
- The General Case

Selected References

Eeckhoudt, Rey, Schlesinger (2007, MS)

Crainich, Eeckhoudt, Le Courtois (2014, JME)

Crainich, Eeckhoudt, Le Courtois (2017, EJOR)

Jouini, Napp, Noceti (2013, JET)

Kimball (1990, Econometrica)

Liu, Meyer (2013, JET)

Ekern (1980, Economics Letters)

Ross (1981, Econometrica)

Previous Results (1D)

What are the conditions on the utility function u such that the introduction of a zero-mean background risk to a portfolio made of a risk-free asset and a risky asset reduces the proportion invested in the risky asset?

A necessary condition for this background risk result is DDRA, or

$$\forall w \quad \frac{\partial}{\partial w}\left(\frac{u'''(w)}{u'(w)}\right) \leqslant 0.$$

A sufficient condition for this background risk result is Ross-DDRA, or

$$\forall t \; \forall w \quad \frac{\partial}{\partial w}\left(\frac{u'''(t+w)}{u'(w)}\right) \leqslant 0.$$

Previous Results (2D)

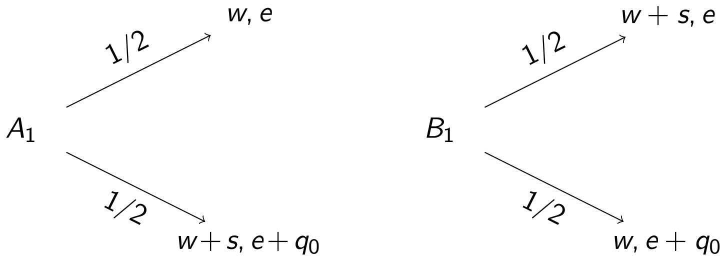
What are the conditions on the utility function u such that the introduction of a zero-mean background risk on health to an agent initially endowed with a portfolio made of a risk-free asset and a risky asset reduces the proportion invested in the risky asset?
A necessary condition for this background risk result is Decreasing Cross Downside Risk Aversion, or DCDRA, where

$$\forall (s,t) \quad \frac{\partial}{\partial s}\left(\frac{u_{122}(s,t)}{u_1(s,t)}\right) \leqslant 0$$

A sufficient condition for this background risk result is Ross-Decreasing Cross Downside Risk Aversion, or Ross-DCDRA, where

$$\forall (s,t,u) \quad \frac{\partial}{\partial s}\left(\frac{u_{122}(s,t+u)}{u_1(s,t)}\right) \leqslant 0$$

A Preliminary Lottery

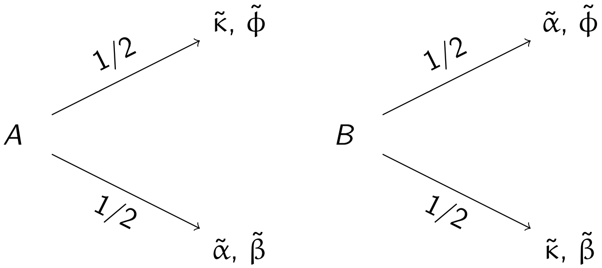


It can be shown that $A_1 \prec B_1$ when individuals are correlation averse, so when $u^{(1,1)} < 0$.

Lotteries

Comparisons

Consider the lotteries A and B described below:



where $\tilde{\kappa}$, $\tilde{\phi}$, $\tilde{\alpha}$, and $\tilde{\beta}$ are four distinct random variables.

We assume that all the partial derivatives of u are non-null. We have the following approximation:

$$E(u(B)) - E(u(A)) \approx \frac{1}{2} \sum_{k=1}^{n_1} \sum_{h=1}^{n_2} \frac{E((\tilde{\alpha}^k - \tilde{\kappa}^k)(\tilde{\phi}^h - \tilde{\beta}^h))}{k!\ h!}\ u_{k,h}(0,0).$$

Let us first generalize a definition from Ekern (1980).

We say that the lottery A has **more** $(n_1, n_2)^{\textbf{th}}$ **degree risk** than the lottery B if

$$\forall (k, h) < (n_1, n_2) \quad E((\tilde{\alpha}^k - \tilde{\kappa}^k)(\tilde{\phi}^h - \tilde{\beta}^h)) = 0$$

and

$$(-1)^{n_1+n_2-1}\ E((\tilde{\alpha}^{n_1} - \tilde{\kappa}^{n_1})(\tilde{\phi}^{n_2} - \tilde{\beta}^{n_2})) > 0.$$

Another generalization of a definition from Ekern (1980) is as follows.

An agent u is $(n_1, n_2)^{\mathbf{th}}$ **degree risk averse** around 0 if and only if $(-1)^{n_1+n_2-1}\ u^{(n_1,n_2)}(0,0) > 0$.

Lotteries

Comparisons

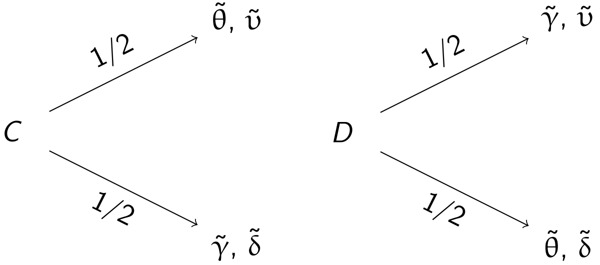
Consider an agent u who is $(n_1, n_2)^{\text{th}}$ degree risk averse around 0. This agent has the choice between two lotteries A and B, where the lottery A has more $(n_1, n_2)^{\text{th}}$ degree risk than the lottery B. Then, B is preferred to A by this agent and the difference in utilities between the two lotteries is given by

$$E(u(B)) - E(u(A)) \approx \frac{1}{2} \frac{E((\tilde{\alpha}^{n_1} - \tilde{\kappa}^{n_1})(\tilde{\phi}^{n_2} - \tilde{\beta}^{n_2}))}{n_1!\ n_2!}\ u_{n_1,n_2}(0,0) > 0.$$

Lotteries

Substitutions

Let us now introduce two additional lotteries C and D:



where $\tilde{\theta}$, $\tilde{\delta}$, $\tilde{\gamma}$, and $\tilde{\upsilon}$ are independent random variables.

Lotteries

Substitutions

The marginal rate of substitution T_u is defined as follows:

$$T_u = \frac{E(u(B)) - E(u(A))}{E(u(D)) - E(u(C))}$$

When the lottery A has more $(n_1, n_2)^{\text{th}}$ degree risk than the lottery B and the lottery C has more $(m_1, m_2)^{\text{th}}$ degree risk than the lottery D, the substitution rate becomes

$$T_u = \frac{E((\tilde{\alpha}^{n_1} - \tilde{\kappa}^{n_1})(\tilde{\phi}^{n_2} - \tilde{\beta}^{n_2}))}{E((\tilde{\gamma}^{m_1} - \tilde{\theta}^{m_1})(\tilde{\upsilon}^{m_2} - \tilde{\delta}^{m_2}))} \frac{m_1!\ m_2!}{n_1!\ n_2!} \frac{u_{n_1,n_2}(0,0)}{u_{m_1,m_2}(0,0)},$$

An interesting result about preferences is given by

$$E(u(D)) - E(u(C)) > E(u(B)) - E(u(A)) \Leftrightarrow$$
$$\frac{u_{n_1,n_2}(0,0)}{u_{m_1,m_2}(0,0)} < \frac{E((\tilde{\gamma}^{m_1} - \tilde{\theta}^{m_1})(\tilde{\upsilon}^{m_2} - \tilde{\delta}^{m_2}))}{E((\tilde{\alpha}^{n_1} - \tilde{\kappa}^{n_1})(\tilde{\phi}^{n_2} - \tilde{\beta}^{n_2}))} \frac{n_1!\ n_2!}{m_1!\ m_2!}.$$

Lotteries

Substitutions

Our main result for lotteries is as follows. Consider two decision makers u and v. When the lottery A has more $(n_1, n_2)^{\text{th}}$ degree risk than the lottery B and the lottery C has more $(m_1, m_2)^{\text{th}}$ degree risk than the lottery D, then,

$$T_u \geqslant T_v \Leftrightarrow \frac{(-1)^{n_1+n_2-1}\ u^{(n_1,n_2)}(0,0)}{(-1)^{m_1+m_2-1}\ u^{(m_1,m_2)}(0,0)} \geqslant \frac{(-1)^{n_1+n_2-1}\ v^{(n_1,n_2)}(0,0)}{(-1)^{m_1+m_2-1}\ v^{(m_1,m_2)}(0,0)}$$

General Case

Comparisons

We consider a two dimensional random variable whose cumulative distribution function is denoted by F. We construct by successive integrations the function $F^{[k,h]}(.,.)$.

For this purpose, we use the following operations:

$$F^{[k,h]}(x,y) = \int_a^x F^{[k-1,h]}(s,y)ds$$

and

$$F^{[k,h]}(x,y) = \int_a^y F^{[k,h-1]}(x,t)dt,$$

where the initial point is $F^{[1,1]}(x,y) = F(x,y)$.

General Case

Comparisons

A distribution G has **more** $(n_1, n_2)^{\textbf{th}}$ **degree risk** than a distribution F if and only if, for all $(k, h) \leqslant (n_1, n_2)$,

$$G^{[k,h]}(s, b) = F^{[k,h]}(s, b) \quad \forall s \in [a, b],$$

and

$$G^{[k,h]}(b, t) = F^{[k,h]}(b, t) \quad \forall t \in [a, b],$$

and also

$$G^{(n_1,n_2)}(s, t) \geqslant F^{(n_1,n_2)}(s, t) \quad \forall (s, t) \in [a, b]^2.$$

Another generalization of Ekern is as follows. Let $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ be bivariate random vectors that are respectively F-distributed and G-distributed. When G has more $(n_1, n_2)^{\text{th}}$ degree risk than F, then

$$E(X_1^k \ X_2^h) = E(Y_1^k \ Y_2^h) \qquad \forall (k, h) < (n_1, n_2)$$

and

$$(-1)^{n_1+n_2} \ E(X_1^{n_1} \ X_2^{n_2}) \leqslant (-1)^{n_1+n_2} \ E(Y_1^{n_1} \ Y_2^{n_1})$$

General Case

Comparisons

Next, we define:

An agent is $(n_1, n_2)^{\text{th}}$ **degree risk averse** if and only if

$$(-1)^{n_1+n_2-1}\ u^{(n_1,n_2)}(s,t) > 0 \quad \forall (s,t) \in [a,b]^2.$$

General Case

Comparisons

An important intermediate result is as follows:

G has more $(n_1, n_2)^{\text{th}}$ degree risk than F if and only if every $(n_1, n_2)^{\text{th}}$ degree risk averter prefers F to G.

General Case

Substitutions

What is the value of T such that an agent is indifferent between $G(x, y)$ and $(1 - T)F(x, y) + TH(x, y)$?

The answer, denoted by T_u, is the marginal rate of substitution defined by

$$T_u = \frac{\int_a^b \int_a^b u(s,t)\big(dF(s,t) - dG(s,t)\big)}{\int_a^b \int_a^b u(s,t)\big(dF(s,t) - dH(s,t)\big)}$$

u is $((n_1, n_2)/(m_1, m_2))^{\text{th}}$ **degree more risk averse** than v if, for all $(s, t) \in [a, b]^2$,

$$\frac{(-1)^{n_1+n_2-1}\ u^{(n_1,n_2)}(s,t)}{(-1)^{m_1+m_2-1}\ u^{(m_1,m_2)}(s,t)} \geqslant \frac{(-1)^{n_1+n_2-1}\ v^{(n_1,n_2)}(s,t)}{(-1)^{m_1+m_2-1}\ v^{(m_1,m_2)}(s,t)}$$

General Case

Substitutions

u is $((n_1, n_2)/(m_1, m_2))^{\text{th}}$ **degree Ross more risk averse** than v if, for all $(s, t) \in [a, b]^2$ and for all $(w, z) \in [a, b]^2$,

$$\frac{(-1)^{n_1+n_2-1} \; u^{(n_1,n_2)}(s, t)}{(-1)^{m_1+m_2-1} \; u^{(m_1,m_2)}(w, z)} \geqslant \frac{(-1)^{n_1+n_2-1} \; v^{(n_1,n_2)}(s, t)}{(-1)^{m_1+m_2-1} \; v^{(m_1,m_2)}(w, z)}$$

General Case

Substitutions

Let two agents u and v that are each both $(n_1, n_2)^{\text{th}}$ degree risk averse and $(m_1, m_2)^{\text{th}}$ degree risk averse. These are equivalent:

(i) u is $((n_1, n_2)/(m_1, m_2))^{\text{th}}$ degree Ross more risk averse than v on $[a, b]^2$, so that there exists $\lambda > 0$ such that $\frac{u^{(n_1,n_2)}(s,t)}{v^{(n_1,n_2)}(s,t)} \geqslant \lambda \geqslant \frac{u^{(m_1,m_2)}(w,z)}{v^{(m_1,m_2)}(w,z)}$ for all $(s, t) \in [a, b]^2$ and $(w, z) \in [a, b]^2$.

(ii) There exist $\lambda > 0$ and $\phi : [a, b]^2 \to \mathbb{R}$ such that $u = \lambda v + \phi$ and such that $(-1)^{m_1+m_2-1}\phi^{(m_1,m_2)}(s, t) \leqslant 0$ and $(-1)^{n_1+n_2-1}\phi^{(n_1,n_2)}(s, t) \geqslant 0$ for all $(s, t) \in [a, b]^2$.

(iii) $T_u \geqslant T_v$ for all F, G, and H such that G has more $(n_1, n_2)^{\text{th}}$ degree risk than F and H has more $(m_1, m_2)^{\text{th}}$ degree risk than F

Conclusion

Our papers and their slides can be downloaded from

www.olivierlecourtois.com

or from

www.ssrn.com