Credit Risk and Solvency Capital Requirements

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Outline of the Talk

- Credit Modeling
- Credit Benchmarking
- Dynamic Portfolios
- Solvency Capital Requirements

Risk-Neutral Credit Transitions

The risk-neutral rating transition matrix T^Q be defined by:

$$T^Q(t,t') = \begin{bmatrix} q_{1,1}(t,t') & \dots & q_{1,K}(t,t') & q_{1,K+1}(t,t') \\ \vdots & & \vdots & \vdots \\ q_{K,1}(t,t') & \dots & q_{K,K}(t,t') & q_{K,K+1}(t,t') \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

It depends on the risk-neutral generator Λ^Q as follows:

$$T^Q(0,t) = e^{\int_0^t \Lambda^Q(s)ds}.$$

Risk-Premium Adjustment Factors

The risk-premium adjustment factor Π is the matrix that satisfies:

$$\Lambda^Q(t) = \Pi(t)\Lambda^P$$

where Λ^P is the historical generator that corresponds to the historical rating transition matrix T^P.

The article examines other types of risk-premium adjustment factors. For brevity, these slides concentrate on Π.

Risk-Neutral Default Probabilities

The risk-neutral default probability can be deduced using the previous definitions. It satisfies:

$$Q(\tau < T_N | \kappa_0 = k) = q_{k,K+1}(0, T_N) = \left[e^{\int_0^{T_N} \Pi(s) \Lambda^P ds} \right]_{k,K+1}$$

Bond Portfolio Value

The total value of a bond portfolio can be expressed as

$$V = \sum_{k=1}^{K} \sum_{j=1}^{M_k} V^{j,k},$$

where $V^{j,k}$ is the value of the j^{th} bond of rating k. For each bond, we have:

$$V^{j,k} = \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) \left(R^{j,k} + (1 - R^{j,k})(1 - Q(\tau^{j,k} < T_i)) \right),$$

where $C_i^{j,k}$ is a coupon or principal payment at time T_i, $R^{j,k}$ is the recovery rate and $\tau^{j,k}$ is the default time of the j^{th} bond of rating k.

Bond Portfolio Value

Therefore, we have:

$$V = \sum_{k=1}^{K} \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) \left(R^{j,k} + (1 - R^{j,k})(1 - Q(\tau^{j,k} < T_i)) \right),$$

or, if the matrix $\Lambda(.)$ is stationary, $V =$

$$\sum_{k=1}^{K} \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) - \sum_{k=1}^{K} \sum_{j=1}^{M_k} (1 - R^{j,k}) \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) \left[e^{\Pi \Lambda^P T_i} \right]_{k, K+1}$$

First Simplified Case

When the matrix Π is driven by a unique constant number π such that $\Pi = \mathrm{diag}(\pi, \ldots, \pi, 1)$, the bond portfolio value becomes $V =$

$$\sum_{k=1}^{K} \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) - \sum_{k=1}^{K} \sum_{j=1}^{M_k} (1 - R^{j,k}) \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) \left[e^{\mathrm{diag}(\pi, \ldots, \pi, 1) \Lambda^P T_i} \right]_{k,K+1}$$

Determination of π

Denoting by $\hat{V}$, the quoted value of the bond portfolio, π can be obtained as the solution to the equation

$$V = \hat{V}$$

or, as the quantity that verifies:

$$\min_{\pi} \sum_{k=1}^{K} (V_k - \hat{V}_k)^2.$$

where V_k is the value of the bonds whose rating is k.

We use the first method in the remainder of the article.

Second Simplified Case

When the matrix Π is driven by K constant numbers $\pi_{k=1,\dots,K}$ such that $\Pi = \text{diag}(\pi_1, \dots, \pi_K, 1)$, the bond portfolio value becomes:

$$V = \sum_{k=1}^{K} \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i)$$

$$- \sum_{k=1}^{K} \sum_{j=1}^{M_k} (1 - R^{j,k}) \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) \left[e^{\text{diag}(\pi_1,\dots,\pi_K,1)\Lambda^P T_i} \right]_{k,K+1},$$

Determination of $\pi_{k=1,\dots,K}$

The numbers $\pi_{k=1,\dots,K}$ can be obtained by solving the system of equations

$$\left\{ \forall k = 1 : K \quad \hat{V}^k = V^k \right\},$$

where

$$V^k = \sum_{j=1}^{M_k} \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) - \sum_{j=1}^{M_k} (1 - R^{j,k}) \sum_{i=1}^{N^{j,k}} C_i^{j,k} P(0, T_i) e^{\mathrm{diag}(\pi_1, \dots, \pi_K, 1) \Lambda^P T_i}]_{k,K+1},$$

so that each V^k depends on the K numbers $\pi_{k=1,\dots,K}$.

Constant Position Framework

Credit SCRs should be computed in the Constant Position Framework: as if the proportions invested in individual assets were never changed by the insurance company.

Consequently, it is not possible in this framework to use the past data of bond portfolios. Instead, the history of bond portfolios should be reconstructed based on current proportions.

Our paper introduces three different approaches to the reconstruction of past bond portfolio data using quoted (Merrill Lynch) bond market indices.

Three Approaches

We introduce the following approaches for reconstructing bond portfolio historical values:

- Approach 1. This approach uses a unique constant π. A unique portfolio is considered.
- Approach 2. This approach uses a unique constant π. The portfolio is split into a sub-portfolio of full recovery bonds and a sub-portfolio of partial recovery bonds.
- Approach 3. This approach uses several constant values $\pi_{k=1,\dots,K}$. A unique portfolio is considered.

Approach 1

The algorithm of the first approach has eight steps:

- Recovering of historical monthly sub-index data.
- Computation of the current value of the portfolio and of its components. Computation of initial portfolio weights.
- Computation of $\pi(0)$.
- Reconstruction of past portfolio returns based on sub-index data and initial portfolio weights.
- Estimation of the average recovery rate of the index. Then, deduction of an initial pseudo-portfolio value whose recovery rate is that of the index.
- Computation of the past values of the pseudo-portfolio.
- Use the pseudo-portfolio to compute $\pi(t)$, $t \leqslant 0$.
- Computation of historical portfolio values.

Approach 2

The algorithm of the second approach is the same as for the first approach, but it is only applied to the portfolio of partial recovery bonds. Full recovery bonds are valued separately similarly to risk-free bonds.

Approach 3

The algorithm of the third approach is the same as for the first approach, but several intermediate risk premium adjustment factors $\pi_{k=1,\ldots,K}$ are used instead of a unique π.

Comparison of Approaches 1, 2 and 3

Issuer	Ranking	Recovery Rate	Maturity	Coupon	Dirty Price	Sensitivity
BEI	AAA	Full	11/10/2016	8%	115.49	1.7
FIN. FONCIER	AAA	Full	29/12/2021	5.62%	121.62	5.9
KFW	AAA	Full	21/01/2019	3.875%	119.08	3.7
GERMANY	AAA	Full	15/08/2023	2%	114.46	8
OAT	AA	Full	25/10/2019	3.75%	117.96	4.5
PROCTER	AA	44%	24/10/2017	5.125%	114.91	2.7
STATOIL	AA	40%	10/09/2025	2.875%	117.38	9.3
COMMONWEALTH	AA	40%	10/11/2016	4.25%	108.09	1.8
AIRBUS GP FIN.	A	55%	12/08/2016	4.625%	108.47	1.6
AIRBUS GROUP FIN.	A	55%	25/09/2018	5.5%	120.48	3.4
AIR LIQ.FIN	A	56%	15/10/2021	2.125%	110.02	6.3
CREDIT AGRICOLE	A	61%	22/12/2024	3%	101.01	8.4
PIRELLI INTER	BBB	64%	18/11/2019	1.75%	101.10	4.4
SEB	BBB	65%	03/06/2016	4.5%	107.71	1.4
VEOLIA	BBB	65%	24/05/2022	5.125%	131.83	6.3
URENCO FINANCE	BBB	60%	02/12/2024	2.375%	101.33	8.6

Table: Bond dataset (iso-weighted) as of 31/12/2014

Comparison of Approaches 1, 2 and 3

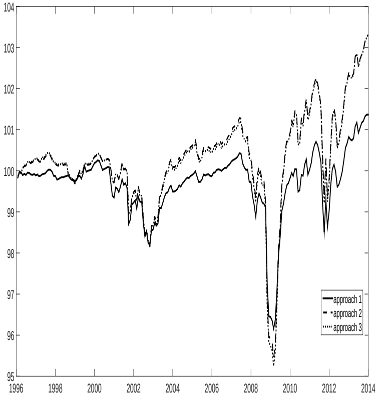

Figure: Credit Portfolio Reconstructed History, Approaches 1, 2, and 3

Which Model?

An important question is the choice of a dynamic model for the representation of $\Pi(.)$.

Specifically, should $\Pi(.)$ be represented by an autoregressive process, such as a CIR process?

To address this question, we show in the next slides the autocorrelations of daily and monthly returns (and squared returns).

Autocorrelations

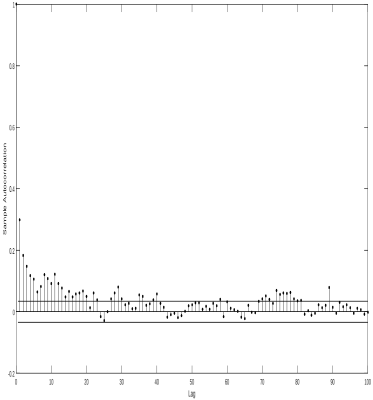

Figure: Autocorrelations of Daily Returns

Autocorrelations

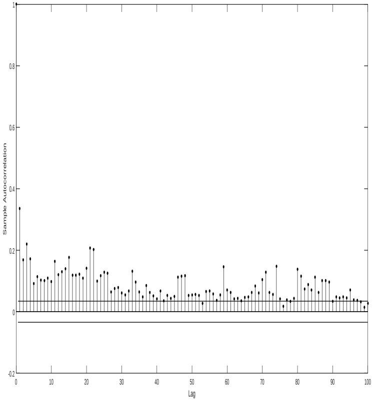

Figure: Autocorrelations of Daily Squared Returns

Autocorrelations

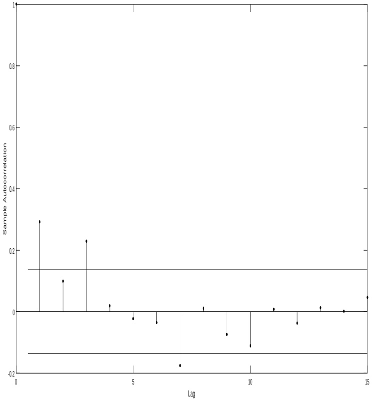

Figure: Autocorrelations of Monthly Returns

Autocorrelations

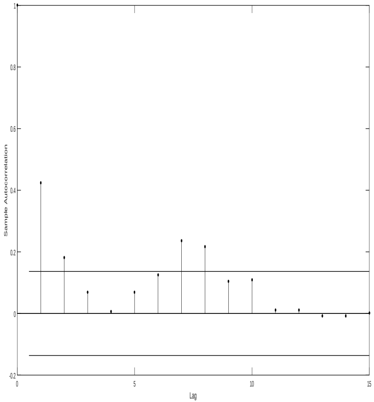

Figure: Autocorrelations of Monthly Squared Returns

Model Choice

From the values of autocorrelations, we conclude that it is not necessary to take into account dependence as long as the discretization frequency of $\Pi()$ is sufficiently low (monthly or less than monthly).

In practice, we only need to simulate $\Pi(.)$ yearly, so that it is not necessary to model the dependence of increments.

We use an independent stationary increments process for modeling the fluctuations of $\Pi(.)$. We choose the Generalized Pareto Distribution (GPD) for modeling the marginal distribution that underlies the process.

Estimation

Our goal is now to estimate the GPD parameters in

$$G_{u,\xi,\sigma}(x) = 1 - \left(1 + \frac{\xi}{\sigma}(x - u)\right)^{-\frac{1}{\xi}}$$

for all $x \geqslant u$ and assuming $\xi \neq 0$.

For this purpose, we use an algorithm that is described in Le Courtois and Walter (2014) and that is a mixture of the Hill and Hosking and Wallis methods.

Estimation

The results of our estimation are as follows:

	$\hat{u}^n$	$\hat{\sigma}^n$	$\hat{\xi}^n$	$\hat{u}^p$	$\hat{\sigma}^p$	$\hat{\xi}^p$
App. 1	0.00319	0.00132	0.61115	0.00411	0.00126	0.48838
App. 2	0.00016	0.00134	0.49649	0.00561	0.00214	0.45648
App. 3	0.00013	0.00129	0.48936	0.00483	0.00176	0.46078

Table: GPD parameters.

Validation

We use several methods for validating our estimation:

- Maximum Likelihood
- POT Graphs
- Lorenz Curves
- Gini Coefficients
- Kolmogorov-Smirnov Test

For brevity, we only report here the results of the last two methods.

Validation (Gini coefficients)

Lorenz curves represent the accumulation of excess returns (ranked in decreasing order and whose total sum is normalized to one for convenience) as a function of the proportion of such excess returns in the data.

Gini coefficients measure the ratio of the surface between the Lorenz curve and the first diagonal of the square to the surface of the half square.

Validation (Gini coefficients)

We compute Gini coefficients, where n and p respectively denote negative and positive contributions, and where t and e respectively denote the GPD (theoretical) and data (empirical) values.

	G_t^n	G_e^n	G_t^p	G_e^p
Approach 1	72%	66%	66%	58%
Approach 2	67%	65%	65%	53%
Approach 3	66%	65%	65%	56%

Table: Gini coefficients.

The greatest confirmation that the data can be modeled using the Generalized Pareto Distribution is achieved for losses in approaches 2 and 3, which is our main situation of interest.

Validation (KS test)

We also perform a Kolmogorov-Smirnov test, where θ denotes the statistics of the test and θ^c its critical value. We also compute the associated p-values.

	θ^n	θ_c^n	p-value^n	θ^p	θ_c^p	p-value^p
Approach 1	0.11	0.29	0.96	0.20	0.34	0.52
Approach 2	0.07	0.14	0.79	0.21	0.39	0.65
Approach 3	0.07	0.14	0.80	0.17	0.35	0.74

Table: Kolmogorov-Smirnov statistics and p-values.

All the results of the table confirm that we cannot reject the GPD hypothesis.

SCRs (simplified definition)

For simplicity, we introduce a definition of an SCR that does not look at the effect of credit shocks on the best estimate value of the liabilities of the insurance firm.

We define:

$$SCR = \frac{\mathcal{A}_0 - \mathcal{A}_{99.5\%}}{\mathcal{A}_0}$$

where $\mathcal{A}_0$ is the current value of the credit sensitive assets and $\mathcal{A}_{99.5\%}$ is the 50^{th} worst value of these assets out of $10,000$ scenarios.

SCRs (computation)

We compute Credit SCRs under the standard formula:

$$SCR^{StandardFormula} = a + b\ (S - c)$$

where a, b and c are given in *ad hoc* tables and S is the sensitivity of bonds.

We also compute credit SCRs under six different internal models: $SCR^{i=1,2,3}$ and $SCR^{i=1,2,3;GPD}$. The first three models rank the data reconstructed using approaches 1 to 3. The last three models compute quantiles from the GPD fit of the data reconstructed using approaches 1 to 3.

SCRs (results)

	Standard Formula	1	1+GPD	2	2+GPD	3	3+GPD
Total Portfolio	5.5%	5.2%	6.1%	6.9%	7.9%	6.7%	7.2%
Sub Portfolio AAA	1.4%	0%	0%	0%	0%	0%	0%
Sub Portfolio AA	3.2%	6.4%	7.1%	11.5%	9.1%	9.1%	8.5%
Sub Portfolio A	6.1%	8.3%	8.9%	10.6%	11.4%	12.5%	10.6%
Sub Portfolio BBB	11.8%	7.1%	9%	8.9%	11.1%	8%	10.4%

Table: Credit SCRs.

SCRs (interpretation)

The main conclusions that stem from the table are:

- Internal models all give smaller SCRs than the standard formula for AAA and BBB bonds.
- The opposite observation holds for AA and A bonds.
- This is a consequence of the construction of our portfolio (high recovery low ranking bond picking)
- Internal models predict a larger total SCR than the standard formula, but the opposite could be observed with a differently constructed portfolio.
- Approaches 2 and 3, which are more accurate than approach 1, are also more conservative.
- The effect of smoothing is not predictable.