q-Credibility

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Outline of the Talk

- Parametric approach
- Non-parametric approach
- Semi-parametric approach

Parametric Approach

We want to solve the following program:

$$\min_{\alpha_{0,q},\{\alpha_i\},\{\beta_i\}} E\left(\left[\alpha_{0,q}+\sum_{i=1}^{n}\alpha_i X_i+\sum_{i=1}^{n}\beta_i X_i^2-E(X_{n+1}|\theta)\right]^2\right).$$

that extends credibility theory to a quadratic setting.

Classic Notation

For hypothetical means: $\mu(\theta) = E(X|\theta)$, $\mu = E(\mu(\theta))$.
For the process variance: $v(\theta) = \text{Var}(X|\theta)$, $v = E(v(\theta))$.
For the variance of hypothetical means: $a = \text{Var}(\mu(\theta))$.
Then,

$$\text{Cov}(X_i, X_k) = a, \qquad \forall i \neq k,$$

and,

$$\text{Cov}(X_i, X_i) = \text{Var}(X_i) = a + v.$$

Additional Notation

By analogy, we define

$$\mathrm{Cov}(X_i^2, X_k) = b, \qquad \forall i \neq k,$$

and

$$\mathrm{Cov}(X_i^2, X_i) = b + g,$$

and also

$$\mathrm{Cov}(X_i^2, X_k^2) = c, \qquad \forall i \neq k,$$

and

$$\mathrm{Cov}(X_i^2, X_i^2) = \mathrm{Var}(X_i^2) = c + h.$$

q-Credibility Premium

We have:

$$P_q = \alpha_{0,q} + Z_q \bar{X} + Y_q \overline{X^2},$$

where

$$\alpha_{0,q} = \mu(1 - Z_q) - Y_q(\mu^2 + a + v),$$

and

$$Z_q = \frac{n\left[a(nc+h) - b(nb+g)\right]}{(na+v)(nc+h) - (nb+g)^2},$$

and

$$Y_q = \frac{n(bv - ag)}{(na+v)(nc+h) - (nb+g)^2}.$$

q-Credibility Premium

We also have:

$$P_q = \mu + Z_q(\bar{X} - \mu) + Y_q(\overline{X^2} - (\mu^2 + v + a)).$$

Remark

When $b = g = 0$,

without any constraint on c or h,

then $Z_q = \frac{na}{na+v} = \frac{n}{n+\frac{v}{a}}$, $Y_q = 0$ and $\alpha_{0,q} = \mu(1 - Z_q)$,

so we recover the classic credibility case.

Mean Square Error

The Mean Square Error is computed as follows:

$$\mathrm{MSE}_q = E\left(\left[\alpha_{0,q}^* + \sum_{i=1}^{n} \alpha_i^* X_i + \sum_{i=1}^{n} \beta_i^* X_i^2 - E(X_{n+1}|\theta)\right]^2\right),$$

yielding

$$\mathrm{MSE}_q = \frac{nv(ac - b^2) + a(hv - g^2)}{n^2(ac - b^2) + n(ah + cv - 2bg) + hv - g^2}.$$

In the classic context, the formula reduces to

$$\mathrm{MSE} = \frac{va}{na + v}.$$

The relative gain in MSE is measured by the quantity:

$$\kappa = \frac{\mathrm{MSE} - \mathrm{MSE}_q}{\mathrm{MSE}}.$$

Conditional Poisson Case

Denote θ as λ and X as N and assume that N conditional on λ is Poisson distributed. $\mu = v = E(\lambda)$ and $a = \mathrm{Var}(\lambda)$ in classic credibility theory. Then,

$$b = a + E(\lambda^3) - E(\lambda^2)\ E(\lambda),$$

and

$$g = E(\lambda) + 2E(\lambda^2),$$

and also

$$c = 2b - a + \mathrm{Var}(\lambda^2),$$

and

$$h = E(\lambda) + 6E(\lambda^2) + 4E(\lambda^3).$$

Poisson-Gamma Case

q-credibility reduces to classic credibility, with

$$Y_q = 0,$$

and

$$Z_q = Z,$$

and also

$$\alpha_{0,q} = \alpha_0 = \mu(1 - Z).$$

Poisson-Single Pareto Case

Let λ be single Pareto-distributed with parameters $(\eta > 4, \chi)$. We know that $\mu = v = \frac{\eta\chi}{\eta-1}$ and $a = \frac{\eta\chi^2}{\eta-2} - \left(\frac{\eta\chi}{\eta-1}\right)^2$. Then,

$g = \frac{\eta\chi}{\eta-1} + 2\frac{\eta\chi^2}{\eta-2}, \qquad b = \frac{\eta\chi^3}{\eta-3} - \frac{\eta\chi^2}{\eta-2}\frac{\eta\chi}{\eta-1},$
and

$$c = 2b - a + \frac{\eta\chi^4}{\eta - 4} - \left(\frac{\eta\chi^2}{\eta - 2}\right)^2,$$

and also

$$h = \frac{\eta\chi}{\eta - 1} + 6\frac{\eta\chi^2}{\eta - 2} + 4\frac{\eta\chi^3}{\eta - 3}.$$

Poisson-Single Pareto Case

Illustration

The parameters of the single Pareto distribution are $\eta = 5$ and $\chi = 4$ and we assume that 5 claims have been observed in the past $n = 2$ years. Then, $\mu = 5$ and $\bar{X} = 2.5$.

According to classic credibility theory, $P = 4$.

Poisson-Single Pareto Case

Illustration

According to the q-credibility approach,

Number of claims distrib.	(3,2)	(4,1)	(5,0)
$\overline{X^2}$	6.5	8.5	12.5
P_q	4.1314	4.1629	4.2259

Poisson-Single Pareto Case

Illustration

We have:

$$\text{MSE} = 1,$$

and

$$\text{MSE}_q = 0.9317,$$

so that

$$\kappa = 6.83\%.$$

Non-Parametric Approach

The classic estimator of expected hypothetical means is

$$\hat{\mu} = \frac{1}{rn} \sum_{i=1}^{r} \sum_{j=1}^{n} X_{ij},$$

and that of expected process variance is

$$\hat{v} = \frac{1}{r(n-1)} \sum_{i=1}^{r} \sum_{j=1}^{n} (X_{ij} - \bar{X}_i)^2,$$

where $\bar{X}_i = \frac{1}{n} \sum_{j=1}^{n} X_{ij}$ is the empirical mean of past observations for insured i.

Non-Parametric Approach

The estimator of the variance of hypothetical means is

$$\hat{a} = \frac{1}{r-1}\sum_{i=1}^{r}(\bar{X}_i - \bar{X})^2 - \frac{\hat{v}}{n}$$

where $\bar{X}$ is the empirical mean of past observations for all insureds, which is equal to $\hat{\mu}$.

Non-Parametric Approach

The quadratic non-parametric estimators are given as follows.

$$\hat{h} = \frac{1}{r(n-1)} \sum_{i=1}^{r} \sum_{j=1}^{n} \left(X_{ij}^2 - \overline{X_i^2} \right)^2,$$

where $\overline{X_i^2} = \frac{1}{n} \sum_{j=1}^{n} X_{ij}^2$ is the empirical mean of past squared observations for a given insured i.

Non-Parametric Approach

Then,

$$\hat{c} = \frac{1}{r-1} \sum_{i=1}^{r} \left(\overline{X_i^2} - \overline{X^2} \right)^2 - \frac{\hat{h}}{n},$$

where

$$\overline{X^2} = \frac{1}{rn} \sum_{i=1}^{r} \sum_{j=1}^{n} X_{ij}^2$$

is the empirical mean of past squared observations for all insureds.

Non-Parametric Approach

Next,

$$\hat{g} = \frac{1}{r(n-1)} \sum_{i=1}^{r} \sum_{j=1}^{n} (X_{ij}^2 - \overline{X_i^2})(X_{ij} - \bar{X}_i),$$

and

$$\hat{b} = \frac{1}{r-1} \sum_{i=1}^{r} (\overline{X_i^2} - \overline{X^2})(\bar{X}_i - \bar{X}) - \frac{\hat{g}}{n}.$$

Non-Parametric Approach

Illustration

Assume $r = n = 3$ and we have the following data:

$$\mathbf{X} = \begin{pmatrix} 1 & 2 & 6 \\ 1 & 10 & 13 \\ 1 & 1 & 1 \end{pmatrix},$$

where each line is for a zone and each element is a number of yellow submarines observed at each time.

Non-Parametric Approach

Illustration

According to classic credibility theory, the number of yellow submarines observed in each zone is as follows.

$$P_1 \approx 3.3932,$$

and

$$P_2 \approx 6.4274,$$

and

$$P_3 \approx 2.1795.$$

Non-Parametric Approach

Illustration

According to the q-credibility approach, we have:

$$P_{q,1} \approx 2.3890,$$

and

$$P_{q,2} \approx 6.2613,$$

and

$$P_{q,3} \approx 2.2928.$$

Relative changes $\left(\frac{P_{q,i}-P_i}{P_i}\right)_{i=1:3}$ are respectively -29.6%, -2.58%, and 5.2%.

Non-Parametric Approach

Illustration

We have:

$$\mathrm{MSE} = 3.1016,$$

and

$$\mathrm{MSE}_q = 2.7634,$$

so that

$$\kappa = 10.9\%.$$

Semi-Parametric Approach

Situation where the distribution of X conditionally on θ is known. Here, we assume this conditional distribution to be of the Poisson type and X is denoted by N.

Each n_i describes the number of insureds for which i claims occurred.

Semi-Parametric Approach

In the classic setting, we have:

$$\tilde{\mu} = \tilde{v} = \sum_{i=0}^{+\infty} i \frac{n_i}{\sum_{i=0}^{+\infty} n_i} = \frac{1}{M} \sum_{i=0}^{+\infty} i n_i,$$

where $M = \sum_{i=0}^{+\infty} n_i$. Then,

$$\tilde{a} = \frac{\sum_{i=0}^{+\infty} (i - \tilde{\mu})^2 n_i}{\sum_{i=0}^{+\infty} n_i - 1} - \tilde{v} = \frac{1}{M-1} \sum_{i=0}^{+\infty} (i - \tilde{\mu})^2 n_i - \tilde{v}.$$

Semi-Parametric Approach

In the q-credibility setting, we have:

$$\tilde{g} = \frac{1}{M} \sum_{i=0}^{+\infty} (2i^2 - i) n_i,$$

and

$$\tilde{b} = \frac{1}{M-1} \sum_{i=0}^{+\infty} \left(i^2 - \frac{1}{M} \sum_{i=0}^{+\infty} i^2 n_i \right) \left(i - \frac{1}{M} \sum_{i=0}^{+\infty} i n_i \right) n_i - \tilde{g},$$

Semi-Parametric Approach

Further,

$$\tilde{h} = \frac{1}{M} \sum_{i=0}^{+\infty} (4i^3 - 6i^2 + 3i) n_i,$$

and

$$\tilde{c} = \frac{1}{M-1} \sum_{i=0}^{+\infty} \left(i^2 - \frac{1}{M} \sum_{i=0}^{+\infty} i^2 n_i \right)^2 n_i - \tilde{h}.$$

Semi-Parametric Approach

Illustration

Assume we observed

i	n_i
0	560
1	134
2	14
3	2

Semi-Parametric Approach

Illustration

We can estimate the expected future number of claims in the next period for an insured who incurred i claims.

According to classic credibility theory:

$$P = [0.2359 \quad 0.2388 \quad 0.2417 \quad 0.2446].$$

According to the q-credibility approach:

$$P_q = [0.2376 \quad 0.2266 \quad 0.2722 \quad 0.3743].$$

Semi-Parametric Approach

Illustration

We have:

$$\mathrm{MSE} = 0.000681,$$

and

$$\mathrm{MSE}_q = 0.000585,$$

so that

$$\kappa = 14.1\%.$$