The Tempered Multistable Approach and Asset Return Modeling

Olivier Le Courtois
Professor of Finance and Insurance
EM Lyon Business School

Outline of the Talk

1. Bibliography
2. Beyond Lévy Processes
3. Series Representations
4. Dependence
5. Moments and Risk Indicators
6. General Properties
7. Illustration

Bibliography

- Madan and Milne, MF [1991]
- Barndorff-Nielsen, FS [1998]
- Eberlein, Keller and Prause, JoB [1998]
- Carr, Geman, Madan and Yor, JoB [2002]
- Carr, Geman, Madan and Yor, MF [2003]
- Falconer and Lévy-Véhel, JTP [2009]
- Lévy-Véhel and Liu, WP [2013]

Beyond Lévy Processes

stable process -> multistable process :

tail parameter α -> $\alpha(t)$

so that :

Lévy measure $\frac{1}{x^{1+\alpha}}$ -> $\frac{1}{x^{1+\alpha(t)}}$

Beyond Lévy Processes

Multivariate characteristic function of the *independent increments* multistable process :

$$\mathbb{E}\left(e^{i\sum_{j=1}^{d}\theta_j L_{II}(t_j)} \right) = e^{-\int \left|\sum_{j=1}^{d}\theta_j \mathbf{1}_{[0,t_j]}(s)\right|^{\alpha(s)} ds}.$$

This is an additive process.

Beyond Lévy Processes

Multivariate characteristic function of the
field-based multistable process :

$$\mathbb{E}\left(e^{i\sum_{j=1}^{m}\theta_j L_{FB}(t_j)}\right) = e^{-2\int_{[0,T]}\int_0^{+\infty}\sin^2\left(\sum_{j=1}^{m}\theta_j \frac{C_{\alpha(t_j)}^{1/\alpha(t_j)}}{2y^{1/\alpha(t_j)}} 1_{[0,t_j]}(x)\right) dy\, dx}$$

This process has dependent non-stationary increments...
... but still Pareto-like

Beyond Lévy Processes

Univariate characteristic function of the
independent increments **tempered** multistable process :

$$\varphi_{Z_{II}(t)}(\theta) = e^{C\int_0^t \Gamma(-Y(v))\left[(M-i\theta)^{Y(v)}-M^{Y(v)}+(G+i\theta)^{Y(v)}-G^{Y(v)}\right]dv}$$

Beyond Lévy Processes

Univariate characteristic function of the
field-based **tempered** multistable process :

$$\varphi_{Z_{FB}(t)}(\theta) = e^{tC\Gamma(-Y(t))\left[(M-i\theta)^{Y(t)}-M^{Y(t)}+(G+i\theta)^{Y(t)}-G^{Y(t)}\right]}$$

Beyond Lévy Processes

First goal : obtain the multivariate characteristic functions of these processes

Second goal : study their properties and applications in finance

Series Representations

Using Rosiński [2007]'s results, we have the following series representation for the CGMY process when $Y \in (0,1)$:

$$X(t) = \sum_{j=1}^{\infty} \gamma_j \left(\left(\frac{\Gamma_j Y}{2CT} \right)^{-1/Y} \wedge \frac{e_j V_j^{1/Y}}{\frac{M+G}{2} + \gamma_j \frac{M-G}{2}} \right) \mathbb{1}_{(U_j \leq t)}, \; 0 < t \leq T,$$

where Γ_j is an arrival time of a Poisson process with unit arrival rate, U_j is a uniform random variable on $[0, T]$, V_j is a uniform random variable on $[0, 1]$, e_j is a standard exponential random variable, and γ_j is a random variable with distribution $\mathbb{P}(\gamma_j = 1) = \mathbb{P}(\gamma_j = -1) = 1/2$. All these random variables are independent.

Series Representations

Then, when $Y \in [1,2)$:

$$X(t) = \sum_{j=1}^{\infty} \gamma_j \left(\left(\frac{\Gamma_j Y}{2CT} \right)^{-1/Y} \wedge \frac{e_j V_j^{1/Y}}{\frac{M+G}{2} + \gamma_j \frac{M-G}{2}} \right) \mathbb{1}_{(U_j \leq t)} + t\,\eta,\ 0 < t \leq T,$$

where, for $Y \in (1,2)$:

$$\eta = -\ \Gamma(1-Y)\ C\ \left(M^{Y-1} - G^{Y-1} \right)$$

and for $Y = 1$:

$$\eta = (2\kappa + \ln(2T))\ C\ \left(M^{Y-1} - G^{Y-1} \right) + C\ \left(M^{Y-1} \ln(M) - G^{Y-1} \ln(G) \right)$$

and where κ is the Euler constant and $x \wedge y$ stands for $\min(x,y)$.

Series Representations

For the independent increments
tempered multistable process :

$$Z_{II}(t) = \sum_{j=1}^{\infty} \gamma_j \left(\left(\frac{\Gamma_j Y(U_j)}{2CT} \right)^{-1/Y(U_j)} \wedge \frac{e_j V_j^{1/Y(U_j)}}{\frac{M+G}{2} + \gamma_j \frac{M-G}{2}} \right) \mathbb{1}_{(U_j \leq t)}, \ 0 < t \leq T,$$

Series Representations

For the field-based tempered multistable process :

$$Z_{FB}(t) = \sum_{j=1}^{\infty} \gamma_j \left(\left(\frac{\Gamma_j Y(t)}{2CT} \right)^{-1/Y(t)} \wedge \frac{e_j V_j^{1/Y(t)}}{\frac{M+G}{2} + \gamma_j \frac{M-G}{2}} \right) \mathbb{1}_{(U_j \leq t)}, \; 0 < t \leq T,$$

Series Representations

FB-CGMY simulation experiment

with $T = 20$, $C = 1$, $G = 30$, $M = 30$ and $Y(T) = 0.5$:

$j$max	10^3	10^4	10^5	10^6	10^7
$Z_{FB}(T)$	0.83353	0.90376	0.89712	0.89714	0.89714

Dependence

Multivariate char. function of the FB-CGMY process :

$$E\left(e^{i\sum_{k=1}^{K}\theta_k Z_{FB}(t_k)}\right) = e^{\frac{1}{2T}\int_{u=0}^{T}\int_{x=0}^{+\infty}\int_{v=0}^{1}\int_{g=0}^{+\infty} e^{-g}\left(1-e^{i\sum_{k=1}^{K}\theta_k h_M(t_k)\mathbb{1}_{u\le t_k}}\right)dgdvdxdu}$$

$$\times\, e^{\frac{1}{2T}\int_{u=0}^{T}\int_{x=0}^{+\infty}\int_{v=0}^{1}\int_{g=0}^{+\infty} e^{-g}\left(1-e^{-i\sum_{k=1}^{K}\theta_k h_G(t_k)\mathbb{1}_{u\le t_k}}\right)dgdvdxdu}$$

where $h_M(t) = \left(\left(\frac{xY(t)}{2CT}\right)^{-1/Y(t)} \wedge \frac{gv^{1/Y(t)}}{M}\right)$.

Dependence

Let $s < t$. The correlation between the increments $Z_{FB}(t) - Z_{FB}(s)$ and $Z_{FB}(t+\delta) - Z_{FB}(s+\delta)$ satisfies :

$$\rho_{s,t}(\delta) = \frac{\frac{\partial \Psi}{\partial \theta}(0,0)\frac{\partial \Psi}{\partial \eta}(0,0) - \frac{\partial^2 \Psi}{\partial \theta \partial \eta}(0,0)}{\sqrt{\left(\frac{\partial \Psi}{\partial \theta}(0,0)\right)^2 - \frac{\partial^2 \Psi}{\partial \theta^2}(0,0)}\sqrt{\left(\frac{\partial \Psi}{\partial \eta}(0,0)\right)^2 - \frac{\partial^2 \Psi}{\partial \eta^2}(0,0)}}$$

where $\theta = \theta_1$ and $\eta = \theta_2$.

Dependence

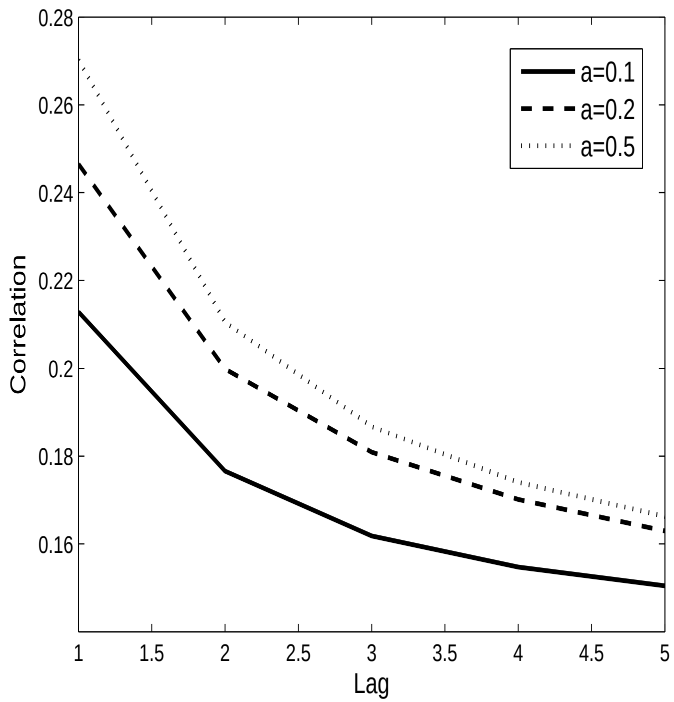
Let us assume :

s	t	M	G	C	T
0	1	60	40	1	20

and :

$$Y(t, a) = 0.1 + 0.8\ (1 - e^{-at})$$

Dependence



0.28
0.26
0.24
0.22
0.2
0.18
0.16
Correlation
1
1.5
2
2.5
3
3.5
4
4.5
5
Lag
a=0.1
a=0.2
a=0.5

Dependence

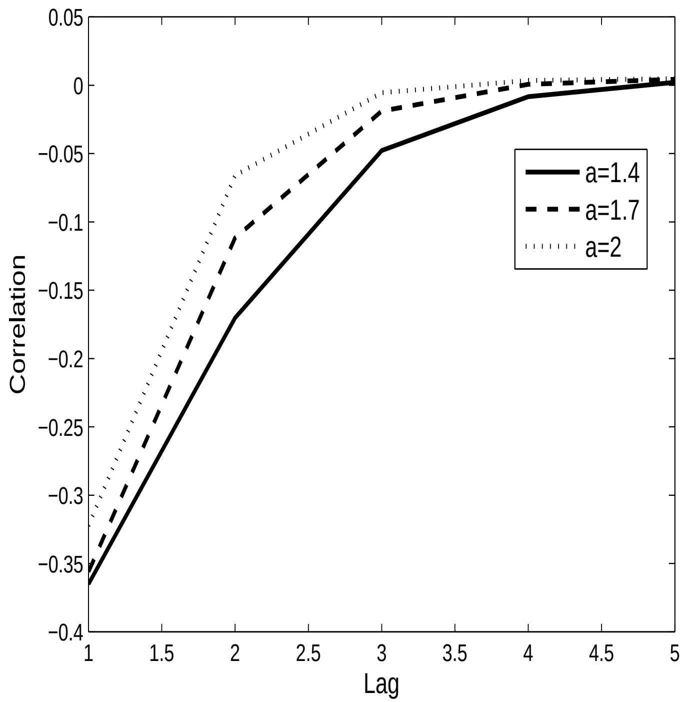
Let us assume :

s	t	M	G	C	T
0	1	50	45	1	20

and :

$$Y(t,a) = 0.1 + 0.8\ e^{-at}$$

Dependence



0.05
0
−0.05
−0.1
−0.15
−0.2
−0.25
−0.3
−0.35
−0.4
Correlation
1
1.5
2
2.5
3
3.5
4
4.5
5
Lag
a=1.4
a=1.7
a=2

Moments and Risk Indicators

The first four moments of the field-based tempered multistable process are given by :

$$\text{Mean }(Z_{FB}(t)) = Ct\Gamma(1 - Y(t))\left(\frac{1}{M^{1-Y(t)}} - \frac{1}{G^{1-Y(t)}}\right),$$

$$\text{Variance }(Z_{FB}(t)) = Ct\Gamma(2 - Y(t))\left(\frac{1}{M^{2-Y(t)}} + \frac{1}{G^{2-Y(t)}}\right),$$

Moments and Risk Indicators

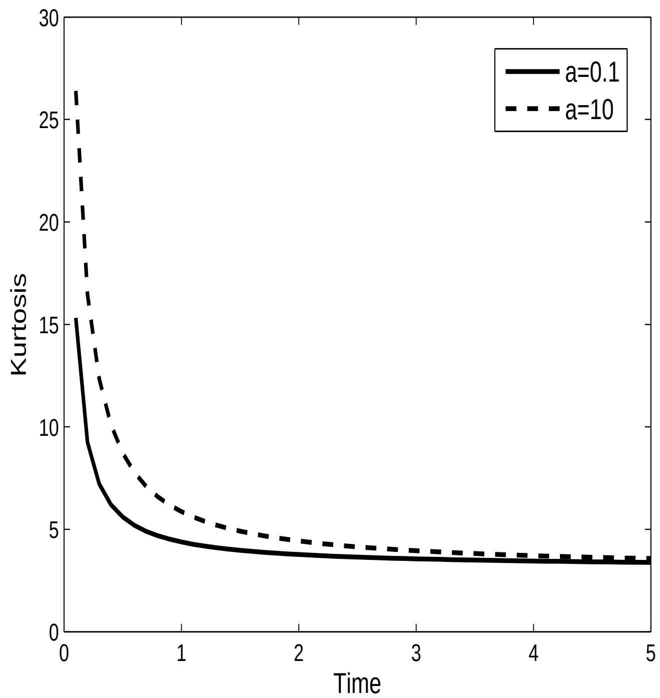
and :

$$\text{Skewness }(Z_{FB}(t)) = \frac{Ct\Gamma(3-Y(t))\left(\frac{1}{M^{3-Y(t)}} - \frac{1}{G^{3-Y(t)}}\right)}{\left(Ct\Gamma(2-Y(t))\left(\frac{1}{M^{2-Y(t)}} + \frac{1}{G^{2-Y(t)}}\right)\right)^{3/2}},$$

$$\text{Kurtosis }(Z_{FB}(t)) = 3 + \frac{Ct\Gamma(4-Y(t))\left(\frac{1}{M^{4-Y(t)}} + \frac{1}{G^{4-Y(t)}}\right)}{\left(Ct\Gamma(2-Y(t))\left(\frac{1}{M^{2-Y(t)}} + \frac{1}{G^{2-Y(t)}}\right)\right)^{2}},$$

and so on at higher orders.

Moments and Risk Indicators



Moments and Risk Indicators

VaR can be computed using for instance :

$$F(x) = \frac{e^{\alpha x}}{2\pi} \int_{-\infty}^{+\infty} e^{iux} \, \frac{\Phi(i\alpha - u)}{\alpha + iu} \, du$$

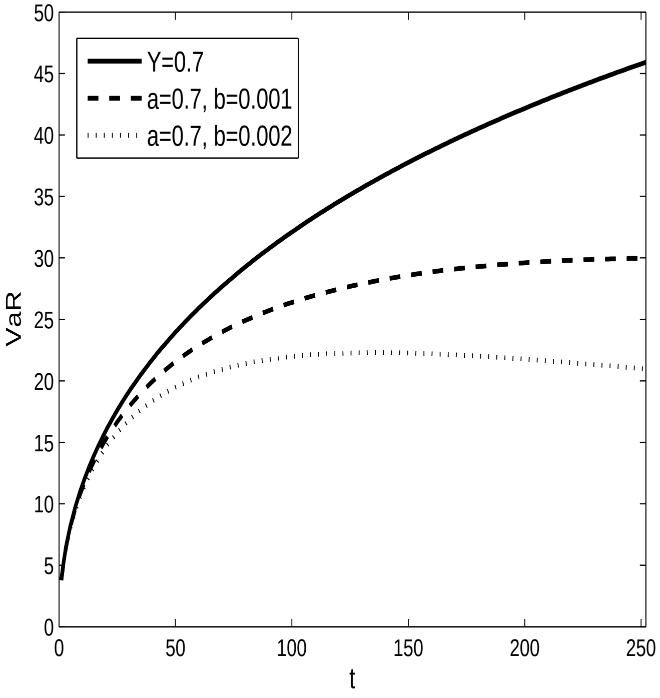
where α can take any positive value.

Moments and Risk Indicators

Let us assume until the end of this presentation :

$$Y(t, a, b) = ae^{-bt}$$

Moments and Risk Indicators



General Properties

The independent increments and the field-based tempered multistable processes are semimartingales.

General Properties

Let us consider an Esscher transform :

$$\left(\frac{dQ}{dP}\right)_t = \frac{e^{\theta X_t}}{E\left(e^{\theta X_t}\right)}$$

The characteristic triplet of an II-CGMY process X under P :

$$\begin{cases} 0, \\ 0, \\ C\frac{e^{Gx}}{x^{1+Y(s)}}\mathbb{1}_{\mathbb{R}-} + C\frac{e^{-Mx}}{x^{1+Y(s)}}\mathbb{1}_{\mathbb{R}+} \end{cases}$$

General Properties

becomes under Q :

$$\begin{cases} C\int_0^t\int_{-1}^0 \frac{e^{(G+\theta)x}-e^{Gx}}{x^{Y(s)}}dxds + C\int_0^t\int_0^1 \frac{e^{-(M-\theta)x}-e^{-Mx}}{x^{Y(s)}}dxds, \\ 0, \\ C\frac{e^{(G+\theta)x}}{x^{1+Y(s)}}\mathbb{1}_{\mathbb{R}-} + C\frac{e^{-(M-\theta)x}}{x^{1+Y(s)}}\mathbb{1}_{\mathbb{R}+} \end{cases}$$

Illustration

We model the logarithmic return of the SP500 Index by the process X defined as follows :

$$X(t) = (\mu - q)t + Z_{FB}(t)$$

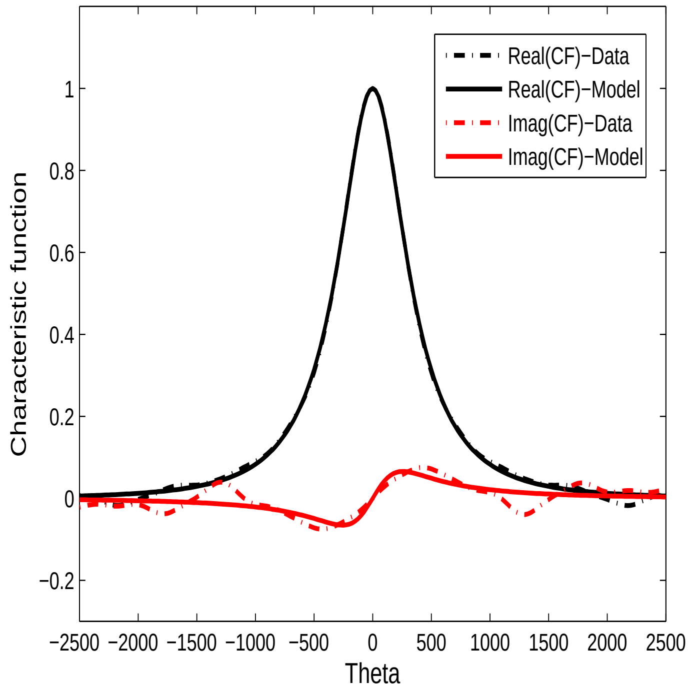
where Z_{FB} is a field-based tempered multistable process. The calibration of the model is carried out as below :

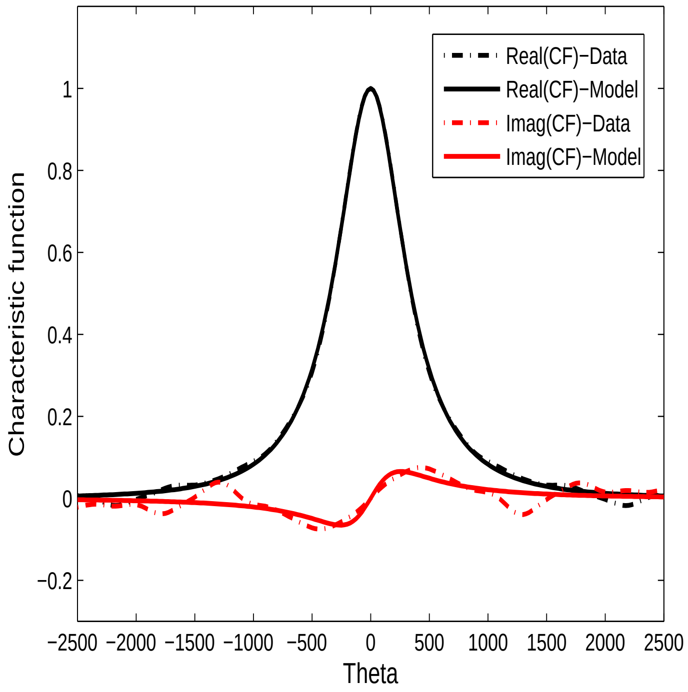
$$\min_{\mu,C,G,M,Y} \sum_{j=1}^{N} \left| E\left(e^{i\theta_j X(t)}\right) - \frac{\sum_{k=1}^{N_b} e^{i\theta_j x_k}}{N_b} \right|^2$$

Illustration

The calibration is performed in two steps. First, we calibrate the model on daily returns and estimate μ, C, G, M and $Y(1, a, b)$. Then, we calibrate the model on ten-day returns and estimate $Y(10, a, b)$. The knowledge of $Y(1, a, b)$ and $Y(10, a, b)$ readily gives a and b.

Illustration





Illustration

The calibrated parameters are :

μ (bp/day)	C	G	M	Y_1	Res1
2.92	0.2261	344.12	310.26	0.2422	0.1061

Y_{10}	Res10	a	b
0.1481	2.3275	0.2558	0.0547

Illustration

The autocorrelations are :

ρ^{data}	ρ^{model}
-4.41%	-3.73%

Illustration

For pricing derivatives, we directly model the stock dynamics in the risk-neutral world as follows :

$$S_t = S_0 e^{(r-q+\omega)t + Z_{FB}(t)}$$

where ω is defined by :

$$e^{-\omega t} = \varphi_{Z_{FB}(t)}(-i) = E_Q\left(e^{Z_{FB}(t)}\right)$$

Illustration

For European call options :

$$C = S_0 e^{-qT}\left(\frac{1}{2} - \frac{i}{2\pi}\int_{\mathbb{R}} \frac{\phi(\theta - i)e^{-i\theta k} - 1}{\theta} d\theta\right)$$

$$- Ke^{-rT}\left(\frac{1}{2} - \frac{i}{2\pi}\int_{\mathbb{R}} \frac{\phi(\theta)e^{-i\theta k} - 1}{\theta} d\theta\right)$$

where :

$$k = \ln\left(\frac{Ke^{-(r-q)T}}{S_0}\right)$$

Illustration

and where :

$$\phi(\theta) = E_Q\left(e^{i\theta \ln\left(\frac{S_T e^{-(r-q)T}}{S_0}\right)} \right)$$

and :

$$\phi(\theta) = e^{i\theta\omega T} E_Q\left(e^{i\theta Z_{FB}(T)}\right) = \frac{\varphi_{Z_{FB}(T)}(\theta)}{\left(\varphi_{Z_{FB}(T)}(-i)\right)^{i\theta}}$$

Illustration

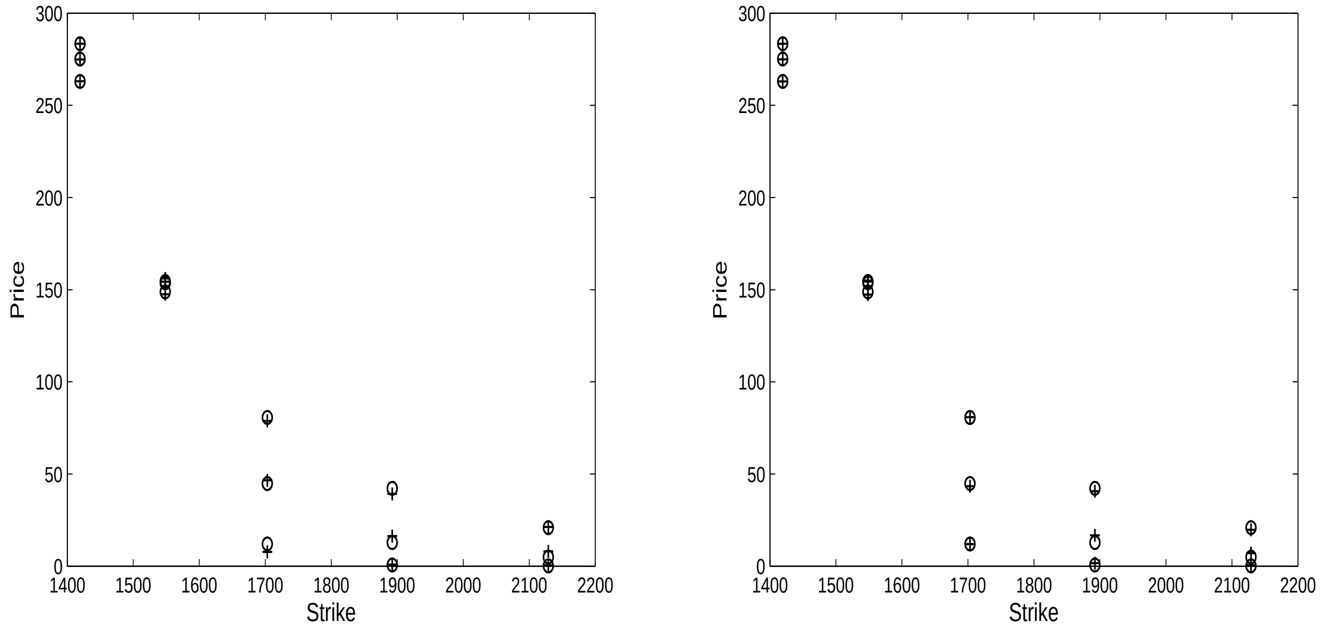
Calibrating on SP500 Index options,
we obtain for the CGMY model :

C	G	M	Y
0.0208	1861	1.4310	1.4508

and for the field-based extended model :

C	G	M	Y^1	Y^2	Y^3
0.2683	24.4760	4.8688	0.9348	0.5270	0.5826

Illustration



Fit of SPX options by CGMY and FBCGMY models

Illustration

Model	APE (%)	AAE	ARPE (%)	RMSE
CGMY	1.4960	0.3395	5.2605	0.9338
FBCGMY	0.9815	0.2227	7.1739	0.2109

Comparison of errors