On Credit and Surrender Risks in Insurance Companies

A joint work by

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Outline of the Talk

1. Bibliography
2. Description of Contracts
3. Modeling Surrenders
4. Some Theoretical Results
5. Case Study
Bibliography

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Bibliography

- Bacinello [NAAJ, 2003]
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### Life Office

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>$E_0 = (1 - \alpha)A_0$</td>
</tr>
<tr>
<td></td>
<td>$L_0 = \alpha A_0$</td>
</tr>
</tbody>
</table>

- $E_0$ = initial equity value
- $L_0$ = initial policyholder investment
Existence of a minimum guaranteed rate $r_g$:

$$L_T^g = L_0 e^{r_g T} \text{ at } T$$

- **Solvency at time** $T$: $A_T \geq L_T^g$
  
  Policyholders receive $L_T^g$

- **Default at time** $T$: $A_T < L_T^g$
  
  Policyholders receive $A_T$
Participating Contracts

–> Participation Bonus

Bonus = $\delta$ times Benefits of the Company, when:

$$A_T > \frac{L^g_T}{\alpha} > L^g_T \quad \left(\alpha = \frac{A_0}{L_0} < 1\right)$$

Assuming no prior bankruptcy, policyholders receive at $T$:

$$\Theta_L(T) =$$

\[
\begin{cases}
  A_T & \text{if } A_T < L^g_T \\
  L^g_T & \text{if } L^g_T \leq A_T \leq \frac{L^g_T}{\alpha} \\
  L^g_T + \delta(\alpha A_T - L^g_T) & \text{if } A_T > \frac{L^g_T}{\alpha}
\end{cases}
\]
The firm pursues its activities until $T$ iff:

$$
\forall t \in [0, T[, \quad A_t > L_0 e^{rgt} \triangleq B_t
$$

Let $\tau$ be the default time

$$
\tau = \inf\{ t \in [0, T] / A_t < B_t \}
$$

In case of prior insolvency, policyholders receive:

$$
\Theta_L(\tau) = A_\tau
$$

(not necessarily equal to $L_0 e^{rg\tau}$)
Modeling Surrenders

Let $N_t^i = 1_{\{\tau^i \leq t\}}$ be the indicator function of surrender of policyholder "i"

The number of cumulative surrenders is:

$$N_t = \sum_{i=1}^{I} N_t^i$$

where $I$ is the initial number of policyholders

We also define by:

$$\bar{A}_t = \frac{A_t}{I - N_t}$$

the assets per remaining policyholder
The asset dynamics is under the risk-neutral probability:

\[ dA_t = A_t - \left( r_t dt + \sigma_t dW_t^Q - \frac{dN_t}{I - N_{t-}} \right) \]

The guarantee \( L_t \) is modeled accordingly:

\[ dL_t = L_t - \left( \rho_g dt - \frac{dN_t}{I - N_{t-}} \right) \]
The interest rate dynamics is under the risk-neutral probability:

\[ dr_t = a (b - r_t) \, dt + \delta_t \, dZ_t^Q \]

The surrender intensity is supposed to follow:

\[ d\lambda_t = c (d - \lambda_t) \, dt + \epsilon_t \, dX_t \]

where we assumed \( W, Z \) and \( X \) correlated.

We also imposed a Gaussian copula on surrenders (common parameter \( \rho \)).
The balance sheet of the firm is, before a surrender event:

<table>
<thead>
<tr>
<th>$t^{-}$</th>
<th>$I - N_{t-}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(I - N_{t-})\bar{A}_t$</td>
<td>$(I - N_{t-})\bar{S}_t$</td>
</tr>
<tr>
<td>$(I - N_{t-})\bar{L}_t$</td>
<td>$(I - N_{t-})\bar{L}_t$</td>
</tr>
</tbody>
</table>
Then, after a surrender event:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$I - N_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(I - N_t)\tilde{A}_t$</td>
<td>$(I - N_t)\tilde{S}_t$</td>
</tr>
<tr>
<td></td>
<td>$(I - N_t)\tilde{L}_t$</td>
</tr>
</tbody>
</table>
Theoretical Results

It can be shown that:

\[ A_t = A_0 \exp \left( \int_0^t \left( r_u - \frac{\sigma_u^2}{2} \right) \, du + \int_0^t \sigma_u \, dW_u^Q \right) \frac{I - N_t}{I} \]

\[ L_t = L_0 e^{\rho g t} \frac{I - N_t}{I} = (I - N_t) \bar{L}_T e^{-\rho g (T-t)} \]

\[ \bar{A}_t = \bar{A}_0 \exp \left( \int_0^t \left( r_u - \frac{\sigma_u^2}{2} \right) \, du + \int_0^t \sigma_u \, dW_u^Q \right) \]
The asset dynamics is under the risk-neutral probability:

$$dA_t = A_t \left( r_t dt + \sigma_t dW^Q_t - \frac{1}{A_t - I - N_t} \beta L_t dN_t \right)$$

The guarantee $L_t$ is modeled accordingly:

$$dL_t = L_t \left( \rho_g dt - \frac{dN_t}{I - N_t} \right)$$
The balance sheet of the firm is, before a surrender event:

\[
\begin{array}{c|c}
 t^- & I - N_{t^-} \\
\hline
(I - N_{t^-}) \bar{A}_{t^-} & (I - N_{t^-}) \tilde{S}_{t^-} \\
& (I - N_{t^-}) \tilde{L}_t \\
\end{array}
\]
Then, after a surrender event:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$I - N_t$</th>
</tr>
</thead>
</table>
|         | $(I - N_t)\bar{A}_t-$  
|         | $+(\bar{A}_t- - \beta\bar{L}_t)$  
|         | $(I - N_t)\bar{S}_t-$  
|         | $+(\bar{A}_t- - \beta\bar{L}_t)$  
|         | $(I - N_t)\bar{L}_t$  |

- A high $\beta$ means surrenders contribute to default
Theoretical Results

It can be shown that:

\[
A_t = A_0 \exp\left( \int_0^t \left( r_u - \frac{\sigma_u^2}{2} \right) du + \int_0^t \sigma_u dW_u \right) \\
- \int_0^t \exp\left( \int_s^t \left( r_u - \frac{\sigma_u^2}{2} \right) du + \int_s^t \sigma_u dW_u \right) \beta L_s dN_s
\]

\[
L_t = L_0 e^{\rho g t \frac{I - N_t}{I}} = (I - N_t) L T e^{-\rho g (T - t)}
\]
An additional assumption

We allow for bubble/acceleration effects as:

\[ \beta_t = \max \left( 1, 1 + \frac{100 - A_t}{100} \right) \]
Example of a Ruin
Ruin Probability
w.r.t. Surrender Correlation
Value of Remaining Contracts

For policyholders remaining with the firm until $T$, and a firm which does not default:

$$EQ\left[(I-N_T)(1-M_T)e^{-\int_0^T r_udu}(\bar{L}_T+\delta(\alpha\bar{A}_T-\bar{L}_T)^+}\right]$$

where $M_t = 1_{\{\tau_d \leq t\}}$ is the indicator of default
Remaining Contracts Value w.r.t. Surrender Intensity
Remaining Contracts Value
w.r.t. Surrender Correlation
Value in Case of Default

The value of the firm upon default is:

\[ E^Q \left[ \int_0^T (I - N_s) e^{-\int_0^s r_u du} \bar{A}_s dM_s \right] \]
Value in Case of Default w.r.t. Surrender Intensity
Value in Case of Default w.r.t. Surrender Correlation
The value of surrendered contracts is:

\[ E^Q \left[ \int_0^T (1 - M_s) e^{-\int_0^s r_u du} \bar{C}_s dN_s \right] \]
Value of Surrendered Contracts w.r.t. Surrender Intensity
Value of Surrendered Contracts w.r.t. Surrender Correlation