On the Market Value of Safety Loadings

APRIA Conference Sidney 2008

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Outline of the Talk

1. Bibliography
2. Standard Participating Contracts
   and the Extended Fortet Method
3. Modified Participating Contracts
   Priced with more Standard Methods
4. An Overall Picture of Guarantees
5. Conclusion

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Standard Participating Contracts and the Extended Fortet Method

Life Office

Assets	Liabilities
A_0	$E_0 = (1 - \alpha)A_0$ $L_0 = \alpha A_0$

- E_0 = initial equity value
- L_0 = initial policyholder investment

Participating Contracts
–> Minimum Guarantee

Existence of a minimum guaranteed rate r_g :

$$L_T^g = L_0 \, e^{r_g T} \quad \text{at } T$$

➠ **Solvency at time** T **:** $\mathbf{A_T \geq L_T^g}$

Policyholders receive L_T^g

➠ **Default at time** T **:** $\mathbf{A_T < L_T^g}$

Policyholders receive A_T

Participating Contracts

−> Participation Bonus

Bonus = δ times Benefits of the Company, when :

$$A_T > \frac{L_T^g}{\alpha} > L_T^g \qquad \left(\alpha = \frac{A_0}{L_0} < 1\right)$$

Assuming *no prior bankruptcy*, policyholders receive at T :

$$\Theta_L(T) = \begin{cases} A_T & \text{if } A_T < L_T^g \\ L_T^g & \text{if } L_T^g \leq A_T \leq \frac{L_T^g}{\alpha} \\ L_T^g + \delta(\alpha A_T - L_T^g) & \text{if } A_T > \frac{L_T^g}{\alpha} \end{cases}$$

Company Early Default

The firm pursues its activities until T iff :

$$\forall t \in [0, T[\quad , \quad A_t > L_0 e^{r_g t} \triangleq B_t$$

Let τ be the default time

$$\tau = \inf\{t \in [0, T] \ / \ A_t < B_t\}$$

In case of prior insolvency, policyholders receive :

$$\Theta_L(\tau) = \ L_0 e^{r_g \tau}$$

Asset Dynamics

The asset dynamics under the risk-neutral probability Q are :

$$\frac{dA_t}{A_t} = r_t dt + \sigma dZ^{\mathsf{Q}}(t)$$

Because a big proportion of the assets are made of bonds, an interest rate model is necessary.

Z^{Q} of the assets will be correlated to Z_1^{Q} of the interest rates ($dZ^{\mathsf{Q}}.dZ_1^{\mathsf{Q}} = \rho dt$).

Stochastic Interest Rates

The dynamics under $\mathbb{Q}$ of the interest rate r
and the zero-coupon bonds $P(t,T)$ are :

$$dr_t = a(\theta - r_t)dt + \nu dZ_1^{\mathbb{Q}}(t)$$

$$\text{and : } \frac{dP(t,T)}{P(t,T)} = r_t dt - \sigma_P(t,T) dZ_1^{\mathbb{Q}}(t)$$

We Assume an Exponential Volatility for the Zero-Coupons :

$$\sigma_P(t,T) = \frac{\nu}{a} \left(1 - e^{-a(T-t)}\right)$$

Contract Valuation

The market value of a standard participating contract is :

$$V_L(0) = \mathbb{E}_Q\Big[e^{-\int_0^T r_s ds}\left[L_T^g + \delta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+\right] \mathbb{1}_{\tau \geq T} + e^{-\int_0^\tau r_s ds} L_0\, e^{r_g \tau} \mathbb{1}_{\tau < T} \Big]$$

This is typically a 2D interest rate/default problem in (r, τ)

To simplify matters, we price the representative term :

$$I = \mathbb{E}_{Q_T}\left[A_T \mathbb{1}_{A_T > \frac{L_T^g}{\alpha}\ ,\ \tau < T}\right]$$

Contract Valuation

We show that :

$$I = e^{r_g T} \int_0^T ds \int_{-\infty}^{+\infty} dr_s \; g(r_s, s) \int_{-\infty}^{+\infty} dr_T \, f_r(r_T \,|\, r_s, \, s, \, l_s) \; \Phi_1\left(\widehat{\mu}_{s,T}; \, \widehat{\Sigma}_{s,T}; \, \frac{L_0}{\alpha}\right)$$

where :

g is the density of (r_τ, τ)

f_r is the Gaussian transition function

Φ_1 is a Gaussian function

l is a return defined by $l_t = \ln(A_t) - r_g t$

$\widehat{\mu}_{s,T}$ and $\widehat{\Sigma}_{s,T}$ are conditional moments of l

Contract Valuation

The previous expression can be discretized as follows :

$$I = \; e^{r_g T} \sum_{j=1}^{n_T} \sum_{i=0}^{n_r} \sum_{k=0}^{n_r} \delta_r f_r(r_k \,|\, r_i,\, t_j,\, l_{t_j}) \;\Phi_1\left(\widehat{\mu}_{t_j,T};\, \widehat{\Sigma}_{t_j,T};\, \frac{L_0}{\alpha}\right) q(i,j)$$

where δ_r is the interest rate step and

$q(i,j)$ is the joint probability of $\tau \in [t_j,\, t_{j+1}]$ and $r \in [r_i, r_{i+1}]$

Finally, one has to solve the recursive equation :

$$q(i,j) = \Phi(\, r_i,\, t_j\,) - \sum_{v=1}^{j-1} \sum_{u=0}^{n_r} q(\, u,\, v\,)\; \Psi(\, r_i,\, t_j \,|\, r_u,\, t_v\,)$$

where Φ and Ψ are completely known.

Modified Participating Contracts Priced with more Standard Methods

A Modified Contract

- differing slightly from the standard one
- in a totally identical framework for A and r
- only the Guaranteed Amount is modified
- Now Indexed on a Risk-Free Zero-Coupon Bond

A New Guarantee

Worth at any time t :

$$l_t^g = \frac{\beta L_0}{P(0,T)} P(t,T) = l_T^g \, P(t,T)$$

where in particular :

$$l_0^g = \beta L_0$$

and

$$l_T^g = \frac{\beta L_0}{P(0,T)}$$

Contract Valuation

The default time becomes :

$$\tau = \inf\left\{ t < T \ / \ A_t < l_t^g \right\}$$

and the contract is priced in market value as :

$$V'(0) = \mathbb{E}_Q \left[e^{-\int_0^T r_s ds} \left(l_T^g + \delta \left(\alpha A_T - l_T^g \right)^+ - \left(l_T^g - A_T \right)^+ \right) \mathbb{1}_{\tau \geq T} + e^{-\int_0^\tau r_s ds} \ l_\tau^g \ \mathbb{1}_{\tau < T} \right]$$

Contract Valuation
Illustration

A typical expression to compute in this setting is :

$$F(T) = \mathsf{Q}_T \left(\tau < T \right)$$

It can be readily shown that :

$$F(T) = \mathsf{Q}_T \left(\inf_{u \in [0,T[} \left(\frac{A_u}{P(u,T)} \right) < l_T^g \right)$$

where l_T^g is a constant

Contract Valuation

The solution of this problem lies in the fact that :

$$\frac{A_u}{P(u,T)} = \frac{A_0}{P(0,T)} e^{N_u - \frac{1}{2}\xi(u)}$$

where the martingale N is defined by :

$$dN_s = (\sigma_P(s,T) + \rho\sigma)\, dZ_1^{Q_T}(s) + \sigma\sqrt{1-\rho^2}\, dZ_2^{Q_T}(s)$$

and its quadratic variation is :

$$\xi(u) = < N >_u = \int_0^u [(\sigma_P(s,T) + \rho\sigma)^2 + \sigma^2(1-\rho^2)]ds$$

Contract Valuation

$$
\begin{aligned}
F(T) &= \mathbb{Q}_T\left(\inf_{u\in[0,T[}\left(\frac{A_u}{P(u,T)}\right) < l_T^g\right)\\
&= \mathbb{Q}_T\left\{\min_{u\in[0,T]}\left(\frac{A_0}{P(0,T)}e^{N_u-\frac{1}{2}\xi(u)}\right) < l_T^g\right\}\\
&= \mathbb{Q}_T\left\{\min_{u\in[0,T]}\left(e^{B_{\xi(u)}-\frac{1}{2}\xi(u)}\right) < \frac{P(0,T)\,l_T^g}{A_0}\right\}\\
&= \mathbb{Q}_T\left\{\min_{s\in[0,\xi(T)]}\left(B_s-\frac{1}{2}s\right) < \ln\left(\beta\alpha\right)\right\}
\end{aligned}
$$

where Dubins-Schwarz is the Key Theorem

An Overall Picture of Guarantees

Market Value of Safety Loadings

Current literature does not pay attention to safety loadings

Safety Loading is an Actuarial Practice
to Protect an Insurance Company

Simply : the higher the immobilized capital per policy,
the lower the ruin probability

Actuaries and insurance regulators want safe companies

We want to tackle this problem from a Finance viewpoint

What the “optional" theory says

Since Merton [1974], for a company like :

Assets	Liabilities
A_0	E_0 (Equity) D_0 (ZC Debt)

Equity is a long call on the assets

Debt is a risk-free ZC bond and a short put on the assets

What the literature mimics

For the last thirty years :

Assets	Liabilities
A_0	E_0 (equity) L_0 (policy)

has been associated to a payoff like :

$$L_T^g + \delta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+$$

(minimum guarantee & bonus & short default put on assets)

What an Insurance Regulator would Want

Such a structure :

Assets	Liabilities
A_0	E_0 (equity) L_0 (policy)

should guarantee to the policyholder the payoff :

$$L_T^g + \delta(\alpha A_T - L_T^g)^+$$

(minimum guarantee & bonus)

Now, the Reality

Companies happen to bankrupt

Life Insurance companies do also bankrupt

−> Perfect guarantees do not exist

Trying to make contracts perfectly safe
is expensive to policyholders...

... furthermore some commercial risk would rise

−> Perfect guarantees would be difficult to create

A Mixed Framework

But :

Policyholders are not (potentially junk-) Bondholders

...Different levels of Risk Aversion, Different Regulations...

−> We construct a mixed framework,

where LI policies are more protected than bonds,
but not fully protected though

(leading to a description of LI policies as hybrid debt)

Framework in Practice

We write for the participating contract's payoff :

$$L_T^g + \delta(\alpha A_T - L_T^g)^+ - (1-\psi) \times (L_T^g - A_T)^+$$

where ψ is the Policyholders'S Immunization degree

ψ is the level of safety :

−> $\psi = 0$ is the Mertonian case (policyholder = bondholder)
−> $\psi = 1$ is for fully safe contracts
−> obviously $0 < \psi < 1$

Also : the higher ψ, the higher protection costs to policyholders

Valuing Policies

−> Assume at this level no early default

The contract can be valued as :

$$V_\psi = \mathbb{E}_{\mathbf{Q}}\left[\, e^{-\int_0^T r_s ds} \left(L_T^g + \delta\ (\alpha A_T - L_T^g)^+ - (1-\psi)\left(L_T^g - A_T\right)^+\right)\,\right]$$

which can be computed for deterministic guarantees

$$(L_t^g = L_T^g e^{-r_g(T-t)})$$

or for stochastic guarantees

$$(L_t^g = L_T^g P(t,T))$$

using the methods described beforehand

Valuing Guarantees

The fully risky contract being worth :

$$V_{\psi=0} = \mathbb{E}_{\mathsf{Q}} \left[\, e^{-\int_0^T r_s ds} \, \left(L_T^g + \delta \, \left(\alpha A_T - L_T^g\right)^+ - \left(L_T^g - A_T \right)^+ \right) \right]$$

The fair price of the guarantee is :

$$V_\psi - V_{\psi=0} = \psi \, \mathbb{E}_{\mathsf{Q}} \left[\, e^{-\int_0^T r_s ds} \, \left(L_T^g - A_T \right)^+ \right]$$

−> Similar Analyses can be done assuming early default/monitoring

Conclusion

A Discussion on the true Nature of Guarantees

A bridge between financial and actuarial theories

Computations done using the Extended Fortet
or Change of Time techniques

Possible Extension to :
Static or Dynamic Management of the underlying Assets