Pricing Derivatives with Barriers in a Stochastic Interest Rate Environment

Carole Bernard (University of Waterloo)
Olivier Le Courtois (E.M. Lyon)
François Quittard-Pinon (University of Lyon)

Introduction

Our aim : pricing **barrier** products
when **interest rates** are **stochastic**.

In order to illustrate our methodology,
we study a particular barrier option : the **Shark Option**.

Yet, our approach is very **general**,
and has applications in lots of other fields.

Relevance

26% of Equity Linked Products currently traded on the American Stock Exchange are **Barrier Products**.

Applications : Extension of Rubinstein and Reiner (1991) Formulas for Barrier Options, Defaultable Bonds, Structured Products, Index Linked Derivatives, Real Options, Stock Options, Portfolio Allocation Market Value of Life Insurance Contracts.

Maturity of traded barrier products can be quite **long**, up to 5 years (among Index Linked Notes).
$\Rightarrow$ **Stochastic Interest Rates**.

Outline of the Talk

- Description of the **Shark Option**
- Underlying and Interest Rate Model
- Two Types of Barriers :
    1. Constant Barrier
    2. Stochastic Barrier

  $\Rightarrow$ Two Valuation Methods
- Numerical Analysis

Shark Options

A Shark Option is :

an **up and out** Barrier Option with a **Rebate**.

The optionholder receives at expiry T :

$$\begin{cases} 1 + \frac{(S_T - S_0)^+}{S_0} & \text{if } \; S_{\max} < H \\ \beta & \text{otherwise} \end{cases}$$

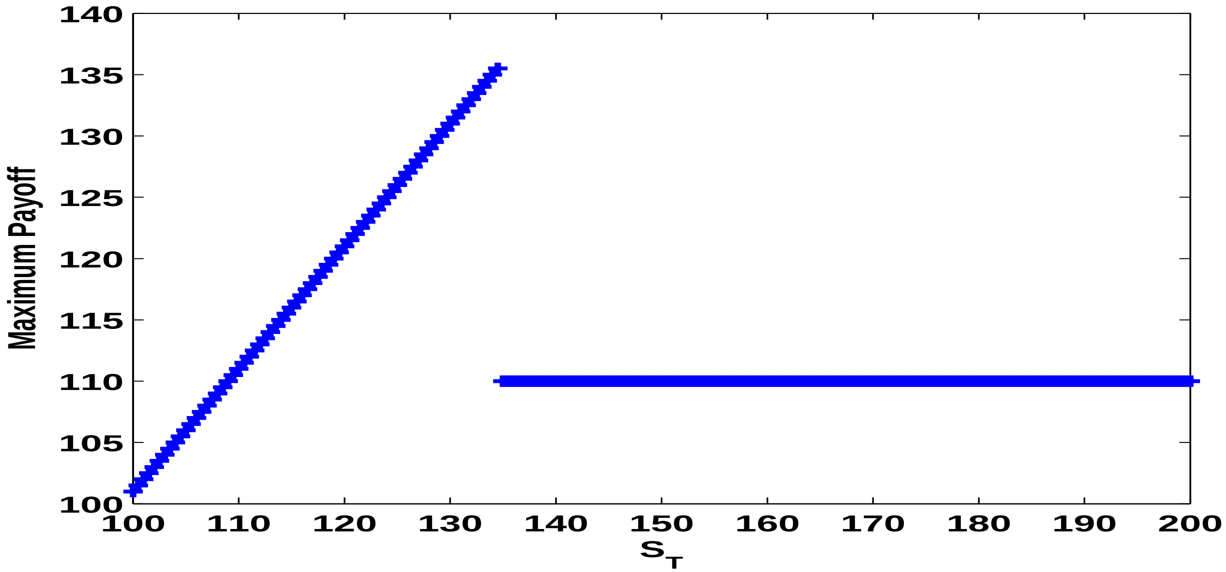
☞ H is the barrier level, β is the rebate.

☞ S_T is the underlying price at time T,

☞ $S_{\max}$ is the maximum of S over $[0, T]$.

Maximum Payoff of a Shark Option

Nominal$= S_0 = 100$, $\beta = 1.1$, $H = 135$



Payoff w.r.t. S_T. ($S_{\max} \geq S_T$) thus this payoff is the maximum the holder might receive.
because one might have $S_{\max} > H$ and $S_T < H$.

Shark Options

Using standard results from arbitrage pricing theory,

we can express the option price (at time 0)

under the **risk-neutral** probability Q :

$$C_0 = E_Q\left(\; e^{-\int_0^T r_s ds} \; \left[\left(1 + \frac{(S_T - S_0)^+}{S_0} \right) \mathbb{1}_{\{S_{\max} \leqslant H\}} + \beta \mathbb{1}_{\{S_{\max} > H\}} \right] \; \right)$$

Interest Rate Modeling

The term structure is given through the **default-free zero-coupon** bonds $P(t,T)$ which dynamics under Q are :

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \sigma_P(t,T) dZ_1^{\mathsf{Q}}(t)$$

We assume an **exponential** volatility :

$$\sigma_P(t,T) = \frac{\nu}{a}\left(1 - e^{-a(T-t)}\right)$$

Underlying Dynamics

The **index** dynamics

under the risk-neutral probability Q are :

$$\frac{dS_t}{S_t} = r_t dt + \sigma dZ^{Q}(t)$$

where Z^{Q} and Z_1^{Q} are correlated Q-Brownian motions.

$(dZ^{Q}.dZ_1^{Q} = \rho dt)$.

Two Steps

☞ **Decorrelation**. Let Z_2^{Q} be independent from Z_1^{Q} :

$$dZ^{\mathsf{Q}}(t) = \rho dZ_1^{\mathsf{Q}}(t) + \sqrt{1-\rho^2} dZ_2^{\mathsf{Q}}(t)$$

☞ **Change of Measure.**

Let Q_T be the T-forward-neutral measure.

From Girsanov theorem, $Z_1^{\mathsf{Q}_T}$ and $Z_2^{\mathsf{Q}_T}$ are independent Q_T-Brownian motions when defined by :

$$dZ_1^{\mathsf{Q}_T} = dZ_1^{\mathsf{Q}} + \sigma_P(t,T)dt \ , \ dZ_2^{\mathsf{Q}_T} = dZ_2^{\mathsf{Q}}$$

Option Valuation at $t = 0$

The Shark option's price (at time 0) is equal to :

$$C_0 = P(0,T)\ E_{Q_T}\left[\ \left(1 + \frac{S_T}{S_0}\right) \mathbb{1}_{\{S_{\text{max}} < H\}} + \beta \mathbb{1}_{\{S_{\text{max}} \geq H\}}\ \right]$$

We obtain the following option **price** :

$$\frac{C_0}{P(0,T)} = \beta\, Q_T\,(\gamma \leqslant T) + Q_T\,(S_T < S_0, \gamma > T) + E_{Q_T}\left[\frac{S_T}{S_0}\mathbb{1}_{\{S_T > S_0, \gamma > T\}}\right]$$

where γ is the first-passage time of S to the level H

Two Types of Barriers

1. **Constant** Barrier : H

   Semi-closed-form Formulae can be obtained.

   Methodology : *Extended Fortet's Approximation*

2. **Discounted** Barrier : $HP(t,T)$

   Closed-form Formulae can be obtained.

Shark Options : Constant Barrier

Problem : We need to know the law of γ,

first passage time of S above H.

- **Longstaff and Schwartz** (1995) use Fortet's results to approximate the density of γ in a problem similar to ours.
- **Collin-Dufresne and Goldstein** (2001) correct the previous method.

First Passage Time Approximate Density

Let us recall the definition of γ :

$$\gamma = \inf\{t \in [0,T] \ / \ S_t < H\}$$

Scheme's Idea : Approximate the density of γ at any time t under Q_T as a piecewise constant function.

– The interval $[0,T]$ is subdivided into n_T subperiods :

$$t_0 = 0, \ ... \ , t_j, \ ... \ , \ t_{n_T} = T$$

– The interest rate is discretized between $r_{\min}$ and $r_{\max}$ into n_r intervals. $r_i = r_{\min} + i\delta_r$ are the discretized values of the interest rate.

The probability of the event $\left\{\gamma \in [t_j,\, t_{j+1}] \text{ with } r \in [r_i, r_{i+1}]\right\}$ is denoted by :

$$\mathbf{q(i,j)}$$

Collin-Dufresne and Goldstein give a recursive formula to compute these probabilities, starting with :

$$q(\, i,\, 0\,) \;=\; \Phi(\, r_i,\, t_0\,)$$

where one first computes $q(\, i,\, 0\,)$ for each i, and then $q(i,j)$ recursively for $j \geq 1$ using :

$$q(i,j) = \Phi(\, r_i,\, t_j\,) - \sum_{v=0}^{j-1} \sum_{u=0}^{n_r} q(\, u,\, v\,)\; \Psi(\, r_i,\, t_j \,|\, r_u,\, t_v\,)$$

where Φ and Ψ are completely known.

Expressions of Φ and Ψ

$$\begin{cases} \Phi(r_t, t) & = f_r(r_t, t|\, l_0, r_0, 0)\, \mathcal{N}\left(\frac{\mu(r_t, l_0, r_0) - h}{\sqrt{\Sigma^2(r_t, l_0, r_0)}}\right) \\ \Psi(r_t, t \,|\, r_s, s) & = f_r(r_t, t \,|\, l_s = h, r_s, s)\, \mathcal{N}\left(\frac{\mu(r_t, l_s{=}h, r_s) - h}{\sqrt{\Sigma^2(r_t, l_s{=}h, r_s)}}\right) \end{cases}$$

where :

* $f_r(r_t, t \,|\, l_s = h, r_s, s) = \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{(r_t - m)^2}{2v}}$, $m = \mathbb{E}[r_t | r_s]$, $v = \mathbb{V}\text{ar}[r_t | r_s]$

* l is defined by : $l_t = \ln S_t$, $\quad h = \ln(H)$,

* μ and Σ are the conditional moments of l.

Shark Options : Constant Barrier

The Shark's price can therefore be expressed as :

$$C_0 = P(0,T)\left[\beta E_1 + E_2\right] + E_3$$

where the three components can be written in terms of such sums :

$$\begin{cases} E_1 & = \sum\limits_{j=0}^{n_T} \sum\limits_{i=0}^{n_r} q(i,j) \\ E_2 & = \mathcal{N}\left(\frac{l_0 - M_T}{\sqrt{V_T}}\right) - \sum\limits_{j=0}^{n_T} \sum\limits_{i=0}^{n_r} \sum\limits_{k=0}^{n_r} \delta_r f_r(r_k \,|\, r_i,\, t_j,\, l_{t_j}) \mathcal{N}\left(\frac{l_0 - \widehat{\mu}_{t_j,T}}{\sqrt{\widehat{\Sigma}^2_{t_j,T}}}\right) q(i,j) \\ \ldots & \end{cases}$$

Shark Options : Stochastic Barrier

From now on,

we suppose the barrier is discounted :

$$\mathbf{D_t = HP(t, T)}$$

$$\gamma = \inf\{t \in [0, T] \ / \ S_t < HP(t, T)\}$$

The barrier is proportional to a zero-coupon bond

($P(t, T)$ is stochastic).

Shark Options : Discounted Barrier

We use time change techniques
in a similar way as **Briys and de Varenne [1997]**
who extended the Black and Cox model [1976]
by considering a stochastic default barrier.

and the following well-known Tools :

- Girsanov Theorem
- Dubins-Schwarz Theorem

The Shark's price can therefore be expressed as :

$$C_0 = P(0,T)\left[\beta E_1 + E_2\right] + E_3$$

where the three components can be written in closed-form :

$$\begin{cases} E_1 & = \mathcal{N}\left(\frac{\ln\left(\frac{S_0}{KP(0,T)}\right)-\frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) + \frac{S_0}{KP(0,T)}\mathcal{N}\left(\frac{\ln\left(\frac{S_0}{KP(0,T)}\right)+\frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) \\ E_2 & = \mathcal{N}\left(\frac{\ln(P(0,T))+\frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) - \frac{S_0}{KP(0,T)}\mathcal{N}\left(\frac{\ln\left(\frac{S_0^2}{K^2P(0,T)}\right)+\frac{\tau(T)}{2}}{\sqrt{\tau(T)}}\right) \\ \ldots \end{cases}$$

where $\tau(T) = \int_0^T \left[(\sigma_P(u,t) + \rho\sigma)^2 + \sigma^2(1-\rho^2)\right] du$

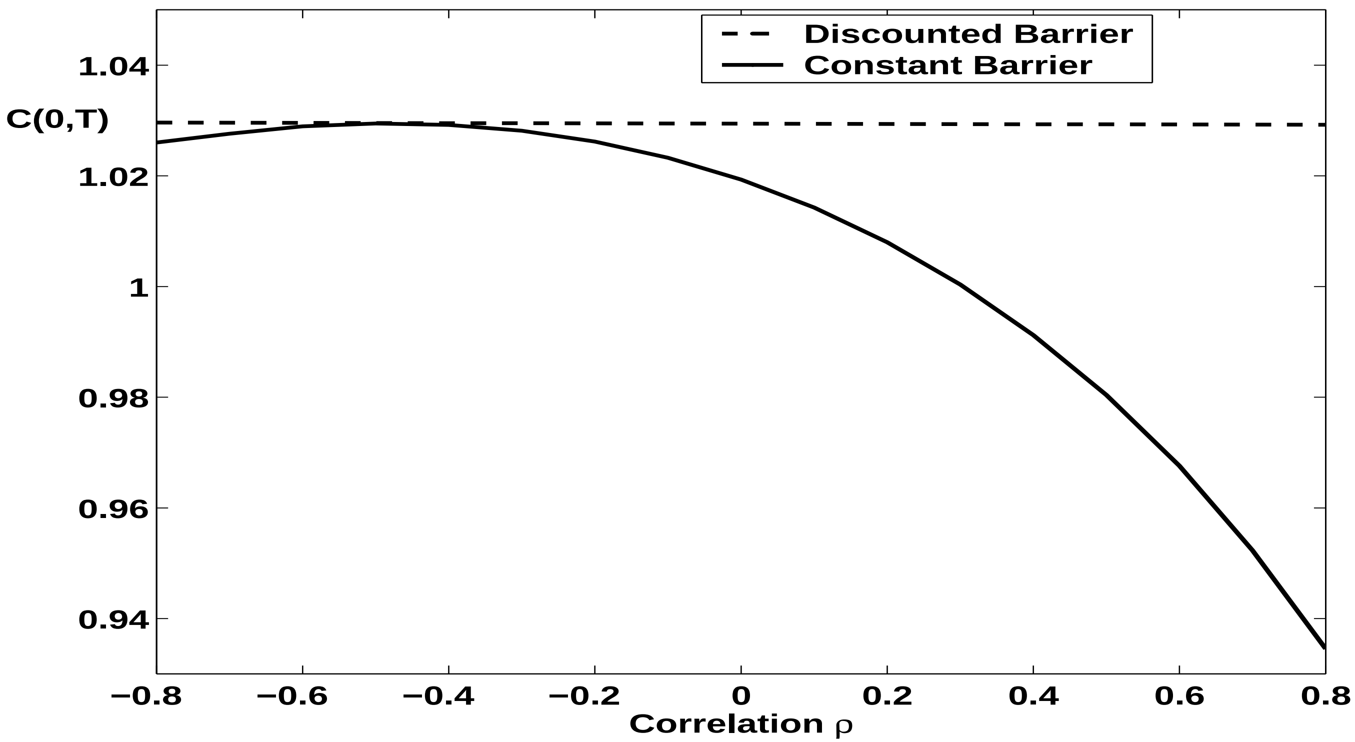
Numerical Analysis

Parameters Chosen Values :

S_0	σ	T	H	β	a	ν	r_0	θ
100	20%	1	135	1.1	0.46	0.007	0.015	0.05

where r_0 and θ give
the initial term structure of interest rates.

Sensitivity to the Correlation ρ



Option Value w.r.t. the correlation ρ.

Conclusion

The aim is to develop a methodology for the pricing of barrier options in closed form and with stochastic interest rates.

When the barrier is **constant**, quasi-closed-form formulae can be found thanks to an Extended Fortet Methodology.

When the derivative's barrier is a **discounted** one, using time change techniques we obtain closed-form formulae.

Conclusion

-> Beyond the chosen example, our article shows how we can price barrier options and compute all their Greeks, under stochastic interest rates.

-> The method yields accurate results (for the prices and Greeks) much more quickly than Monte-Carlo simulations