Management of Pension Funds under Market Jump Risk

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Pension Fund Optimization : Setting

Risk-free asset dynamics :

$$\frac{dS_0(t)}{S_0(t)} = rdt,$$

Risky asset dynamics :

$$\frac{dS(t)}{S(t)} = \mu_L dt + \sigma dW(t) + dL(t),$$

where the jumps of L belong to $]-1, +\infty[$.

Pension Fund Optimization : Setting

Survival probability from t_0
(current time and age at inception)
up to time t :

$$ {}_tp_{t_0} = e^{-\int_{t_0}^{t} \lambda(s)ds}, $$

Gompertz-Makeham assumption :

$$ {}_tp_{t_0} = e^{-\chi(t-t_0)+e^{\frac{t_0-m}{b}}\left(1-e^{\frac{t-t_0}{b}}\right)}. $$

Pension Fund Optimization : Setting

Inflows-outflows to the fund :

$$k(t) = u\mathbb{I}_{t<T} + v(1 - \mathbb{I}_{t<T}),$$

where T is the retirement age.

Retrospective reserve :

$$K(t) = \int_{t_0}^{t} k(s)\, e^{-r(s-t)} ds.$$

Pension Fund Optimization : Setting

Fairness condition :

$$\mathbb{E}_{t_0}^{\mathbb{Q},\tau}\left[\int_{t_0}^{\tau} k\left(t\right) e^{-r(t-t_0)} dt\right] = 0$$

Simplified, under Gompertz-Makeham assumption, as :

$$\frac{v}{u} = \frac{\Gamma\left(-\left(\chi + r\right) b, e^{\frac{t_0 - m}{b}}\right)}{\Gamma\left(-\left(\chi + r\right) b, e^{\frac{T - m}{b}}\right)} - 1,$$

where $\Gamma\left(.,.\right)$ is the upper incomplete Gamma function.

Pension Fund Optimization : Setting

Fund wealth is worth at any time t :

$$R(t) = \theta(t) S(t) + \theta_0(t) S_0(t),$$

After some manipulations, we obtain its dynamics :

$$dR(t) = \left(\frac{(R(t) + \phi(t) K(t)) r}{1 + \phi(t)} + \frac{w(t)(\mu_L - r)}{1 + \phi(t)} + k(t) \right) dt$$
$$+ \frac{w(t)\sigma}{1 + \phi(t)} dW(t) + \frac{w(t)}{1 + \phi(t)} dL(t),$$

where $w(t)$ is the weight invested in the risky asset and ϕ is the financial surplus distributed to pensioners.

Pension Fund Optimization : Setting

We wish to maximise the expected utility of the net wealth $R - K$ at the death time τ of the pensioner :

$$\max_{w(t)} \mathbb{E}_{t_0}^{\tau}\left[e^{-\rho(\tau - t_0)}\frac{1}{1-\delta}\left(R\left(\tau\right) - K\left(\tau\right)\right)^{1-\delta}\right],$$

where ρ is an *ad hoc* discount rate.

Assuming independence of τ and the financial variables, this problem becomes :

$$\max_{w(t)} \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} \lambda\left(t\right) e^{-\int_{t_0}^{t}(\rho+\lambda(s))ds}\frac{1}{1-\delta}\left(R\left(t\right) - K\left(t\right)\right)^{1-\delta} dt\right].$$

Pension Fund Optimization : General Solution

The HJB equation of the pension fund is :

$$0 = \frac{\partial V(R(t), t)}{\partial t} - V(R(t), t)(\lambda(t) + \rho) + \max_{w(t)} \left[\begin{array}{c} \frac{\lambda(t)}{1-\delta}(R(t) - K(t))^{1-\delta} \\ + \frac{\partial V(R(t),t)}{\partial R(t)} \left(\frac{(R(t)+\phi(t)K(t))r}{1+\phi(t)} + \frac{w(t)(\mu_L - r)}{1+\phi(t)} + k(t) \right) \\ + \frac{1}{2} \frac{\partial^2 V(R(t),t)}{\partial R(t)^2} \left(\frac{\sigma w(t)}{1+\phi(t)} \right)^2 \\ + \int_{-1}^{\infty} \left(V\left(R(t) + \frac{w(t)}{1+\phi(t)} z, t \right) - V(R(t), t) \right) \nu(dz) \end{array} \right],$$

where ν is the Lévy measure

and $V(R(t), t)$ is the value function.

Pension Fund Optimization : General Solution

The value function can be cast in the form :

$$V\left(R\left(t\right),t\right)=F\left(t\right)\frac{\lambda\left(t\right)}{1-\delta}\left(R\left(t\right)-K\left(t\right)\right)^{1-\delta},$$

while the first order condition is :

$$\frac{w^{*}\left(t\right)}{1+\phi\left(t\right)}=$$

$$-\frac{\frac{\partial V(R(t),t)}{\partial R(t)}}{\frac{\partial^{2}V(R(t),t)}{\partial R(t)^{2}}}\frac{\mu_{L}-r}{\sigma^{2}}-\frac{1}{\frac{\partial^{2}V(R(t),t)}{\partial R(t)^{2}}}\frac{1}{\sigma^{2}}\int_{-1}^{\infty}\frac{\partial V\left(R\left(t\right)+\frac{w^{*}(t)}{1+\phi(t)}z,t\right)}{\partial\left(R\left(t\right)+\frac{w^{*}(t)}{1+\phi(t)}z\right)}z\nu\left(dz\right).$$

Pension Fund Optimization : General Solution

The renormalized optimal weight $\tilde{w}^*$ defined by :

$$\tilde{w}^* = \frac{w^*(t)}{(1+\phi(t))(R(t)-K(t))},$$

solves :

$$\tilde{w}^* = \frac{1}{\delta}\frac{\mu_L - r}{\sigma^2} + \frac{1}{\delta}\frac{1}{\sigma^2}\int_{-1}^{\infty}(1+\tilde{w}^* z)^{-\delta} z\nu(dz),$$

and is consequently independent of time.

Pension Fund Optimization : General Solution

Assuming jumps pertain to the interval :

$$[-\epsilon_1, \epsilon_2],$$

where ϵ_1 and ϵ_2 are small,
the renormalized optimal weight becomes :

$$\tilde{w}^* = \frac{1}{\delta} \frac{\mu_L - r + \int_{-\epsilon_1}^{\epsilon_2} z \nu\,(dz)}{\sigma^2 + \int_{-\epsilon_1}^{\epsilon_2} z^2 \nu\,(dz)},$$

which is an intuitive extension of the Merton weight.

Pension Fund Optimization : General Solution

As to the function F, it satisfies :

$$F(t) = \int_t^{\infty} e^{-\int_t^s (\lambda(u)+\psi(u))du} ds,$$

where :

$$\psi(t) = \rho - \frac{1-\delta}{1+\phi(t)} r - \tilde{w}^* (\mu_L - r) + \frac{1}{2}\delta(1-\delta)(\tilde{w}^*)^2 \sigma^2 - \int_{-1}^{\infty} \left((1+\tilde{w}^* z)^{1-\delta} - 1 \right) \nu(dz).$$

Pension Fund Optimization : General Solution

Assuming ϕ (and therefore ψ) constant,
and modeling the force of mortality
with the Gompertz-Makeham law,
the expression of F can be simplified as :

$$F(t) = be^{-(\chi+\psi)(m-t)+e^{\frac{t-m}{b}}}\Gamma\left(-(\chi+\psi)b, e^{\frac{t-m}{b}}\right).$$

Pension Fund Optimization : General Solution

Finally, the optimal consumption is worth :

$$\frac{c^{*}(t)}{R(t)-K(t)}=\frac{1}{\int_t^{\infty} e^{-\frac{1}{\delta}\int_t^s (\lambda(u)+\psi)du} ds}$$

A Simple Setting

Assume, with Aït-Sahalia, Cacho-Diaz and Hurd that :

$$\delta = 2,$$

and that :

$$\nu(dz) = \begin{cases} \frac{\lambda^+ dz}{z} & \forall z \ \in]0, +\infty[\\ -\frac{\lambda^- dz}{z} & \forall z \ \in]-1, 0[, \end{cases}$$

then, the renormalized optimal weight solves :

$$\tilde{w}^* = \frac{\mu_L - r}{2\sigma^2} + \frac{1}{2\sigma^2}\left(-\frac{\lambda^-}{1-\tilde{w}^*} + \frac{\lambda^+}{\tilde{w}^*}\right),$$

which is a third degree equation.

A Simple Setting

Assume in addition that there are no positive jumps :

$$\lambda^+ = 0,$$

then, the renormalized optimal weight is equal to :

$$\tilde{w}^* = \frac{\mu_L - r + 2\sigma^2 - \sqrt{\left(\mu_L - r + 2\sigma^2\right)^2 - 8\sigma^2\left(\mu_L - r - \lambda^-\right)}}{4\sigma^2}.$$

Dynamic Frameworks

In most financial frameworks with Lévy processes, stock dynamics are written as :

$$S(t) = S(t_0)\ \ e^{\mu_l (t-t_0) + \sigma W_t + l_t},$$

where l, defined on $\mathbb{R}$, is distinct from L.

We use the above representation for calibration purposes.

Dynamic Frameworks

The Lévy density in the standard exponential framework
can be deduced from the Lévy density
in the stochastic exponential framework as follows :

$$f_l\,(y) = f_L\,(e^y - 1)\,.$$

Conversely :

$$f_L\,(x) = f_l\,(\ln\,(1 + x))\,.$$

Dynamic Frameworks

The Aït-Sahalia, Cacho-Diaz and Hurd [2009] model :

$$f_L(x) = \begin{cases} \frac{\lambda^+}{x} & \forall x \in \,]0, +\infty[\\ -\frac{\lambda^-}{x} & \forall x \in \,]-1, 0[\,, \end{cases}$$

becomes in the standard exponential framework :

$$f_l(y) = \begin{cases} \frac{\lambda^+}{e^y - 1} & \forall y \in \,]0, +\infty[\\ -\frac{\lambda^-}{e^y - 1} & \forall y \in \,]-\infty, 0[\,, \end{cases}$$

with tails :

$f_l(y) \overset{+\infty}{\sim} \lambda^+ e^{-y}$ and $f_l(y) \overset{-\infty}{\sim} \lambda^-$,

where the left one is too thick.

Dynamic Frameworks

We propose an extended stochastic exponential framework :

$$f_L(x) = \begin{cases} \frac{\lambda^+}{x} & \forall x \in \,]0, +\infty[\\ -\frac{\lambda^-}{x} - \lambda^- & \forall x \in \,]-1, 0[\,, \end{cases}$$

which is equivalent to the standard exponential specification :

$$f_l(y) = \begin{cases} \frac{\lambda^+}{e^y - 1} & \forall y \in \,]0, +\infty[\\ \frac{\lambda^-}{e^{-y} - 1} & \forall y \in \,]-\infty, 0[\,, \end{cases}$$

with tails : $f_l(y) \overset{+\infty}{\sim} \lambda^+ e^{-y}$ and $f_l(y) \overset{-\infty}{\sim} \lambda^- e^{y}$.

Dynamic Frameworks

As for the Variance-Gamma model,
defined in the standard exponential framework by :

$$f_l(x) = \begin{cases} \frac{\eta^+ e^{-\alpha x}}{x} & \forall x \in \,]0, +\infty[\\ -\frac{\eta^- e^{\beta x}}{x} & \forall x \in \,]-\infty, 0[\,. \end{cases}$$

it becomes in the stochastic exponential framework :

$$f_L(x) = \begin{cases} \frac{\eta^+}{(1+x)^\alpha \ln(1+x)} & \forall x \in \,]0, +\infty[\\ -\frac{\eta^-(1+x)^\beta}{\ln(1+x)} & \forall x \in \,]-1, 0[\,. \end{cases}$$

Computation of Moments

The moments of the L-process of the extended Aït-Sahalia, Cacho-Diaz and Hurd [2009] framework are all infinite :

$$\mathbb{E}\left(L\left(t\right)\right) = \mathsf{Var}\left(L\left(t\right)\right) = \mathbb{E}\left(\left[L\left(t\right) - \mathbb{E}\left(L\left(t\right)\right)\right]^{n}\right)_{n=3,4} = +\infty.$$

Computation of Moments

The moments of the corresponding l-process are worth :

$$\mathbb{E}\left(l\left(t\right)\right) = t\left(\lambda^{+} - \lambda^{-}\right)\zeta\left(2\right),$$

$$\mathrm{Var}\left(l\left(t\right)\right) = 2\ t\ \left(\lambda^{+} + \lambda^{-}\right)\zeta\left(3\right),$$

$$\mathbb{E}\left(\left[l\left(t\right) - \mathbb{E}\left(l\left(t\right)\right)\right]^{3}\right) = 6\ t\ \left(\lambda^{+} - \lambda^{-}\right)\zeta\left(4\right),$$

$$\mathbb{E}\left(\left[l\left(t\right) - \mathbb{E}\left(l\left(t\right)\right)\right]^{4}\right) = 24\ t\ \left(\lambda^{+} + \lambda^{-}\right)\zeta\left(5\right),$$

where the function ζ is traditionally defined as :

$$\zeta\left(n\right) = \sum_{k=1}^{+\infty} \frac{1}{k^{n}},$$

Computation of Moments

The moments of the l-process of the VG framework are well-known :

$$\mathbb{E}\left(l\left(t\right)\right) = t\ \left(\frac{\eta^{+}}{\alpha} - \frac{\eta^{-}}{\beta}\right),$$

$$\mathrm{Var}\left(l\left(t\right)\right) = t\ \left(\frac{\eta^{+}}{\alpha^{2}} + \frac{\eta^{-}}{\beta^{2}}\right),$$

$$\mathbb{E}\left(\left[l\left(t\right) - \mathbb{E}\left(l\left(t\right)\right)\right]^{3}\right) = 2\ t\ \left(\frac{\eta^{+}}{\alpha^{3}} - \frac{\eta^{-}}{\beta^{3}}\right),$$

$$\mathbb{E}\left(\left[l\left(t\right) - \mathbb{E}\left(l\left(t\right)\right)\right]^{4}\right) = 6\ t\ \left(\frac{\eta^{+}}{\alpha^{4}} + \frac{\eta^{-}}{\beta^{4}}\right).$$

Computation of Moments

The moments of the corresponding L-process are worth :

$$\mathbb{E}\left(L\left(t\right)\right)=\mu t-t\int_{-1}^{0}x\frac{\eta^{-}\left(1+x\right)^{\beta}}{\ln\left(1+x\right)}dx+t\int_{0}^{+\infty}x\frac{\eta^{+}}{\left(1+x\right)^{\alpha}\ln\left(1+x\right)}dx,$$

$$\mathrm{Var}\left(L\left(t\right)\right)=\sigma^{2}t-t\int_{-1}^{0}x^{2}\frac{\eta^{-}\left(1+x\right)^{\beta}}{\ln\left(1+x\right)}dx+t\int_{0}^{+\infty}x^{2}\frac{\eta^{+}}{\left(1+x\right)^{\alpha}\ln\left(1+x\right)}dx,$$

$$m_{3}(L(t))=-t\int_{-1}^{0}x^{3}\frac{\eta^{-}\left(1+x\right)^{\beta}}{\ln\left(1+x\right)}dx+t\int_{0}^{+\infty}x^{3}\frac{\eta^{+}}{\left(1+x\right)^{\alpha}\ln\left(1+x\right)}dx,$$

$$m_{4}(L(t))=-t\int_{-1}^{0}x^{4}\frac{\eta^{-}\left(1+x\right)^{\beta}}{\ln\left(1+x\right)}dx+t\int_{0}^{+\infty}x^{4}\frac{\eta^{+}}{\left(1+x\right)^{\alpha}\ln\left(1+x\right)}dx.$$

Illustration

We fit the S&P500 Index monthly log-returns between February 28th, 1950 and January 30th, 2012.

Three models are considered :

1. the standard mean-variance framework (MV) : logarithmic returns following an arithmetic Brownian motion ;
2. a mixed model (MV+Model I) : logarithmic returns follow an arithmetic Brownian motion + a jump component following model I ;
3. a mixed model (MV+Model II) : logarithmic returns follow an arithmetic Brownian motion + a jump component following model II.

Illustration