Market Value of Life Insurance Contracts under Stochastic Interest Rates and Default Risk

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A Short Actuarial Bibliography...

- Briys and de Varenne [1993, 1997]
- Grosen and Jørgensen [1997, 2000, 2002]
- Tanskanen and Lukkarinen [2003]
- Jørgensen [2004]

...Complemented by some Major References from Finance

- Fortet [1943]
- Merton [1974]
- Heath, Jarrow and Morton [1992]
- Longstaff and Schwartz [1995]
- Collin-Dufresne and Goldstein [2001]

Capital Structure of the Insurance Company

Assets	Liabilities
A_0	$E_0 = (1-\alpha)A_0$ $L_0 = \alpha A_0$

- The life-insurance company has no debt.
- E_0 = initial equity value
- L_0 = initial investment of the policyholders who all possess the same contract.

Simplified Description of the Contract

The policyholders investment L_0 yields the minimum guaranteed rate r_g at contract expiry T.

- **In case of No-default :** $\mathbf{A_T \geq L_0\, e^{r_g T}}$

  Policyholders receive the guaranteed amount at T :

$$L_T^g = L_0\, e^{r_g T}$$

- **In case of Default :** $\mathbf{A_T < L_T^g}$ (Company Insolvency)

  Policyholders receive A_T. Equityholders receive nothing.

A Participating Policy

Policyholders are given a contractual part δ of the benefits of the company when its assets at maturity are sufficiently high :

$$A_T > \frac{L_T^g}{\alpha} \quad \text{where} \quad \alpha < 1.$$

Assuming no prior bankruptcy, policyholders receive at maturity T :

$$\Theta_L(T) = \begin{cases} A_T & \text{si } A_T < L_T^g \\ L_T^g & \text{si } L_T^g \le A_T \le \frac{L_T^g}{\alpha} \\ L_T^g + \delta(\alpha A_T - L_T^g) & \text{si } A_T > \frac{L_T^g}{\alpha} \end{cases}$$

Company Early Default

The firm pursues its activities until T if :

$$\forall t \in [0, T[\quad , \quad A_t > \lambda L_0 e^{r_g t} \triangleq B_t$$

where λ is fixed. Note that the case $\lambda > 1$ is in favour of the policyholders.

Let τ be the default time

$$\tau = \inf\{t \in [0, T] \ / \ A_t < B_t\}$$

In case of prior insolvency, policyholders receive :

$$\Theta_L(\tau) = \begin{cases} L_0 e^{r_g \tau} & \text{si} \quad \lambda \geq 1 \\ \lambda L_0 e^{r_g \tau} & \text{si} \quad \lambda < 1 \end{cases} = \min(\lambda, 1) L_0 e^{r_g \tau}$$

Contract Market Value

Denoting by Q the risk-neutral probability measure, the price of our life insurance contract writes at $t < \tau$:

$$V_L(t) = \mathbb{E}_Q^t \Big[e^{-\int_t^T r_s ds} \left[L_T^g + \delta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+ \right] \mathbb{1}_{\tau \geq T} + e^{-\int_t^\tau r_s ds} \ \mathsf{min}(\lambda, 1) L_\tau^g \ \mathbb{1}_{\tau < T} \Big]$$

This contract can be split up into four simpler subcontracts :

$$V_L = \widehat{GF} + \widehat{BO} - \widehat{PO} + \widehat{LR}$$

- $\widehat{GF}$: the final guarantee
- $\widehat{BO}$: the "bonus option" which is the participating clause
- $\widehat{PO}$: the default put on which policyholders are short.
- $\widehat{LR}$: the rebate paid in case of early default.

Assets Dynamics and Interest Rate Modelling

Exponential Volatility for the Zero-Coupons : $\sigma_P(t,T) = \frac{\nu}{a}\left(1 - e^{-a(T-t)}\right)$

The dynamics under Q of the short interest rate r and the Zero-coupon $P(t,T)$ are :

$$\begin{aligned} dr_t &= a(\theta - r_t)dt + \nu dZ_1^{Q}(t) \\ \text{and} \quad \frac{dP(t,T)}{P(t,T)} &= r_t dt - \sigma_P(t,T) dZ_1^{Q}(t) \end{aligned}$$

The assets follow : $\frac{dA_t}{A_t} = r_t dt + \sigma dZ^{Q}(t)$ where Z^{Q} and Z_1^{Q} are correlated Q-Brownian motions. ($dZ^{Q}.dZ_1^{Q} = \rho dt$).

Decorrelation

Let us now consider a Brownian motion $Z_2^{\mathbb{Q}}$ independent from $Z_1^{\mathbb{Q}}$. The Brownian motion $Z^{\mathbb{Q}}$ can be expressed as

$$dZ^{\mathbb{Q}}(t) = \rho dZ_1^{\mathbb{Q}}(t) + \sqrt{1-\rho^2} dZ_2^{\mathbb{Q}}(t)$$

In this way we decorrelate the interest rate risk from the firm assets risk. The assets dynamics then writes :

$$\frac{dA_t}{A_t} = r_t dt + \sigma \left(\rho dZ_1^{\mathbb{Q}}(t) + \sqrt{1-\rho^2} dZ_2^{\mathbb{Q}}(t) \right)$$

Forward-Neutral Expressions

Let Q_T be the T-forward-neutral measure. From Girsanov theorem, $Z_1^{\mathsf{Q}_T}$ and $Z_2^{\mathsf{Q}_T}$ are independent Q_T-Brownian motions.

$$dZ_1^{\mathsf{Q}_T} = dZ_1^{\mathsf{Q}} + \sigma_P(t,T)dt \ , \ dZ_2^{\mathsf{Q}_T} = dZ_2^{\mathsf{Q}}$$

Under Q_T the prices $P(t,T)$ and A_t follow the stochastic differential equations :

$$\frac{dP(t,T)}{P(t,T)} = (r_t + \sigma_P^2(t,T))dt - \sigma_P(t,T)dZ_1^{\mathsf{Q}_T}$$

and

$$\frac{dA_t}{A_t} = (r_t - \sigma\rho\sigma_P(t,T))dt + \sigma\left(\rho dZ_1^{\mathsf{Q}_T} + \sqrt{1-\rho^2}dZ_2^{\mathsf{Q}_T}\right)$$

Contract Valuation at $t = 0$

After changing the probability measure, we have in the Forward-Neutral universe :

$$V_L(0) = P(0,T)\,(\; GF + BO - PO + LR \;)$$

where

$$\begin{cases} GF &= \; L_T^g \;\; (1 - E_1) \\ \\ BO &= \alpha\delta(E_7 - E_2) - \delta L_T^g (E_8 - E_3) \\ \\ PO &= \; L_T^g \;\; (E_9 - E_4) - E_{10} + E_5 \\ \\ LR &= \; \min(\lambda, 1) L_0 \;\, E_6 \end{cases}$$

with the following quantities that remain to be computed :

$$E_1 = \mathbb{Q}_T\left[\tau < T\right]$$

$$E_2 = \mathbb{E}_{\mathbb{Q}_T}\left[A_T \mathbb{1}_{\left\{A_T > \frac{L_T^g}{\alpha}\ ,\ \tau < T\right\}}\right]$$

$$E_3 = \mathbb{Q}_T\left[A_T > \frac{L_T^g}{\alpha}\ ,\ \tau < T\right]$$

$$E_4 = \mathbb{Q}_T[A_T < L_T^g\ ,\ \tau < T]$$

$$E_5 = \mathbb{E}_{\mathbb{Q}_T}\left[A_T \mathbb{1}_{A_T < L_T^g} \mathbb{1}_{\tau < T}\right]$$

$$E_6 = \mathbb{E}_{\mathbb{Q}_T}\left[e^{r_g \tau} \mathbb{1}_{\tau < T}\right]$$

$$E_7 = \mathbb{E}_{\mathbb{Q}_T}\left[A_T \mathbb{1}_{A_T > \frac{L_T^g}{\alpha}}\right]$$

$$E_8 = \mathbb{Q}_T\left[A_T > \frac{L_T^g}{\alpha}\right]$$

$$E_9 = \mathbb{Q}_T[A_T < L_T^g]$$

$$E_{10} = \mathbb{E}_{\mathbb{Q}_T}\left[A_T \mathbb{1}_{A_T < L_T^g}\right]$$

Methodology :

Longstaff and Schwartz Approximation

Problem : We need to know the law of τ, first passage time of the assets beyond the default-triggering barrier.

- Longstaff and Schwartz (1995) use Fortet's result to approximate the density of τ in a problem similar to ours.

- Collin-Dufresne and Goldstein (2001) give a correction to the previous method to take properly into account the stochastic feature of the interest rates.

Rationale

Let us remember the proper expression for τ

$$\tau = \inf\{t \in [0,T] \ / \ A_t < \lambda L_0 e^{r_g t}\}$$

Idea : Approximate the density of τ at time t under Q_T as a piecewise constant function.

– The interval $[0,T]$ is subdivided into n_T subperiods.
– The interest rate is discretized between $r_{\min}$ and $r_{\max}$ into n_r intervals.

$t_j = j\delta_t$ and $r_i = r_{\min} + i\delta_r$ are the discretized values of time and interest rate.

The probability of the event $\tau \in [t_j, t_{j+1}]$ with $r \in [r_i, r_{i+1}]$ expresses as : $\mathbf{q(i,j)}$.

Collin-Dufresne and Goldstein give a recursive formula for these probabilities :

$$q(\, i,\, 1\,) \;=\; \sum_{u=0}^{n_r} q(\, u,\, 1\,)\; \Psi(\, r_i,\, t_1 \,|\, r_u,\, t_1\,)$$

One would first compute $q(\, i,\, 1\,)$ for each i, and then $q(i,j)$ recursively for $j \geq 2$ using :

$$q(i,j) = \Phi(\, r_i,\, t_j\,) - \sum_{v=1}^{j-1} \sum_{u=0}^{n_r} q(\, u,\, v\,)\; \Psi(\, r_i,\, t_j \,|\, r_u,\, t_v\,)$$

where Φ and Ψ are completely known.

Expressions of Φ and Ψ

$$\mathcal{L}\left(l_t|\mathcal{F}_s, r_t\right) = \mathcal{G}auss\left(\mu(\,r_t,\, l_s,\, r_s\,), \Sigma^2(\,r_t,\, l_s,\, r_s\,)\right)$$

let $\mathcal{N}$ be the cumulative fonction of the $\mathcal{G}auss(0,1)$ law, then :

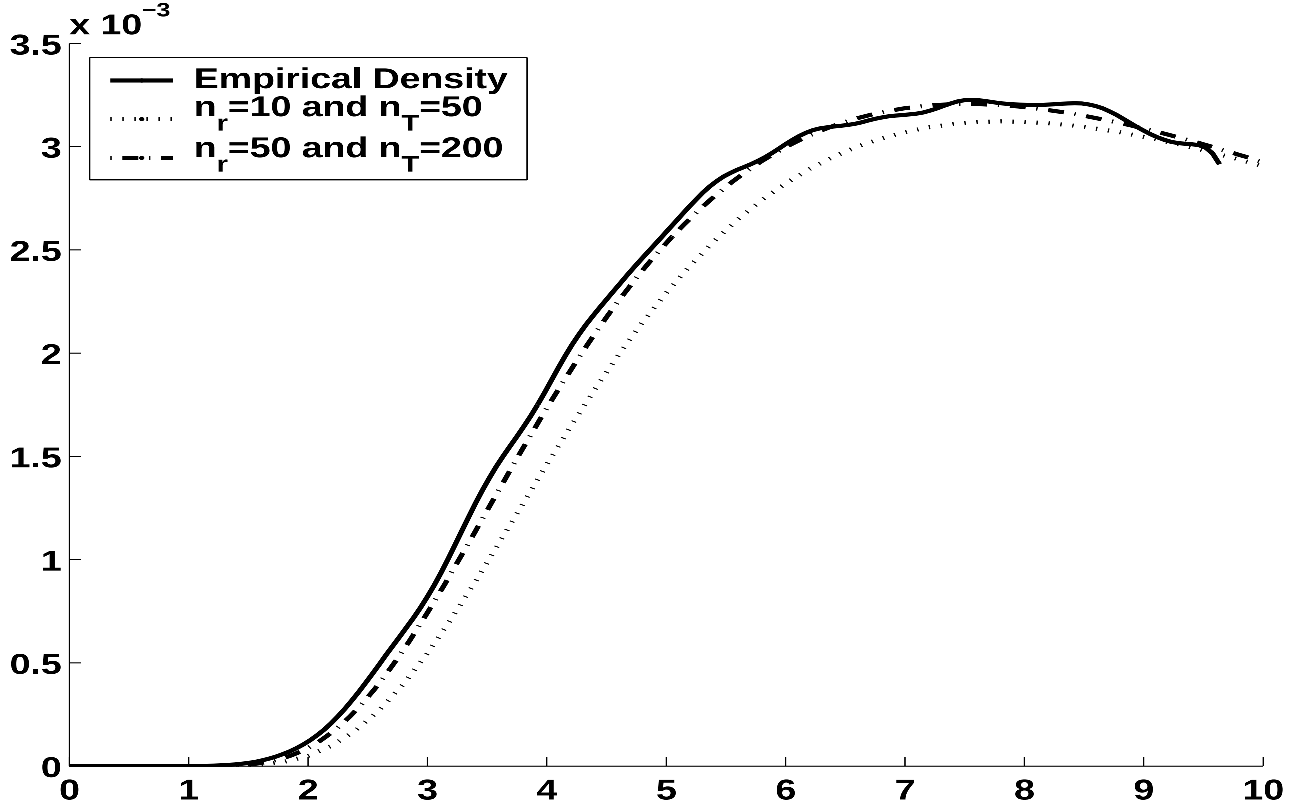
$$\Phi(\,r_t,\, t\,) \quad = \quad f_r(\,r_t,\, t|\; l_0,\, r_0,\, 0)\; \mathcal{N}\left(\frac{h - \mu(\,r_t,\, l_0,\, r_0\,)}{\sqrt{\Sigma^2(\,r_t,\, l_0,\, r_0\,)}}\right)$$

$$\Psi(\,r_t,\, t\,|\,r_s,\, s\,) \;\; = \;\; f_r(\,r_t,\, t\,|\; l_s = h,\, r_s,\, s)\; \mathcal{N}\left(\frac{h - \mu(\,r_t,\, l_s = h,\, r_s\,)}{\sqrt{\Sigma^2(\,r_t,\, l_s = h,\, r_s\,)}}\right)$$

where :

$$f_r(\,r_t,\, t\,|\; l_s = h,\, r_s,\, s) = \frac{1}{\sqrt{2\pi v}}\; e^{-\frac{(rt-m)^2}{2v}} \;\;,\;\; m = \mathbb{E}[r_t|r_s] \;,\;\; v = \mathbb{V}\mathrm{ar}[r_t|r_s]$$

Empirical Density and Fortet's Approximate Density



Computation of the E_i depending on τ

Now, each E_i can be computed easily even if it depends on τ. We detail how to valuate E_2 for instance ; its exact expression is :

$$E_2 = \mathbb{E}_{\mathbb{Q}_T}\left[A_T \mathbb{1}_{\left\{A_T > \frac{L_T^g}{\alpha}\ ,\ \tau < T\right\}}\right]$$

Then, using conditional laws we obtain :

$$E_2 = e^{r_g T} \int_0^T ds \int_{-\infty}^{+\infty} dr_s\ g(r_s, s) \mathbb{E}_{\mathbb{Q}_T}\left[e^{l_T} \mathbb{1}_{\left\{l_T > \ln\left(\frac{L_0}{\alpha}\right)\right\}} \mid l_s = h,\ r_s,\ s,\ \tau = s\right]$$

As $\mathcal{L}\left(l_t | \mathcal{F}_s, r_t\right) = \mathcal{G}auss\left(\mu, \Sigma^2\right)$ and as we know the transition density of r : f_r we can compute E_2 discretizing the integrals.

Let X be a random variable with law $\mathcal{N}(m, \sigma^2)$, we denote

$$\Phi_1(m; \sigma; a) \;=\; \mathbb{E}[e^X \mathbb{1}_{e^X > a}] \;=\; \exp\left(m + \frac{\sigma^2}{2}\right) \mathcal{N}\left(\frac{m + \sigma^2 - \ln(a)}{\sigma}\right)$$

E_2 then admits the simpler expression :

$$E_2 = e^{r_g T} \int_0^T ds \int_{-\infty}^{+\infty} dr_s \; g(r_s, s) \int_{-\infty}^{+\infty} dr_T \, f_r(r_T \,|\, r_s, \; s, \; l_s) \; \Phi_1\left(\widehat{\mu}_{s,T}; \; \widehat{\Sigma}_{s,T}; \; \frac{L_0}{\alpha}\right)$$

The extended Fortet's approximation of E_2 writes :

$$E_2 = \; e^{r_g T} \sum_{j=1}^{n_T} \sum_{i=0}^{n_r} \sum_{k=0}^{n_r} \delta_r f_r(r_k \,|\, r_i, \; t_j, \; l_{t_j}) \; \Phi_1\left(\widehat{\mu}_{t_j,T}; \; \widehat{\Sigma}_{t_j,T}; \; \frac{L_0}{\alpha}\right) q(i,j)$$

Numerical Analysis

We set our parameter range according as :

A_0	a	ν	θ	r_0	ρ	σ	T	λ	α
100	0.4	0.008	0.06	0.03	- 0.02	0.1	10	0.8	0.7

$L_0 = \alpha A_0 = 70$

Contract Maturity : 10 years

Numerical Results

Extended Fortet	GF	BO	PO	LR	Contract	Time
$n_T = 200, \; n_r = 50$	28.11	89.05	0.09	1.27	60.9967	2 min
$n_T = 500, \; n_r = 50$	28.11	89.03	0.09	1.29	69.9996	10 min

Monte-Carlo	GF	BO	PO	LR	Contract	Time
$step = 1/12$	28.10	89.28	0.14	1.30	70.1108	15 min
$step = 1/52$	28.11	89.14	0.13	1.31	70.0451	1h20
$step = 1/365$	28.14	89.07	0.13	1.30	70.0201	1 jour

Contract Fair Value

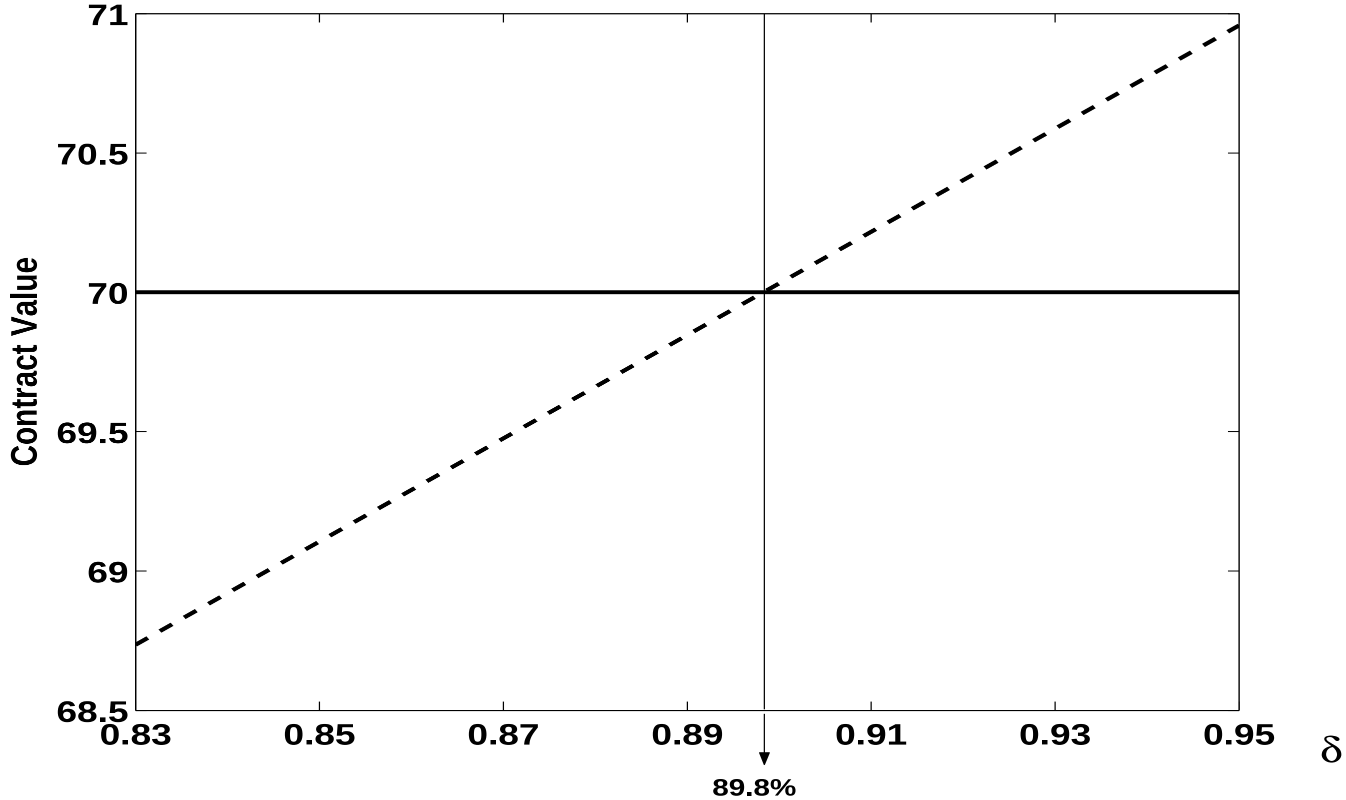
Definition : The initial investment of policyholders $L_0 = \alpha A_0$ must be equal to the contract market value at $t = 0$.

The Parameters :

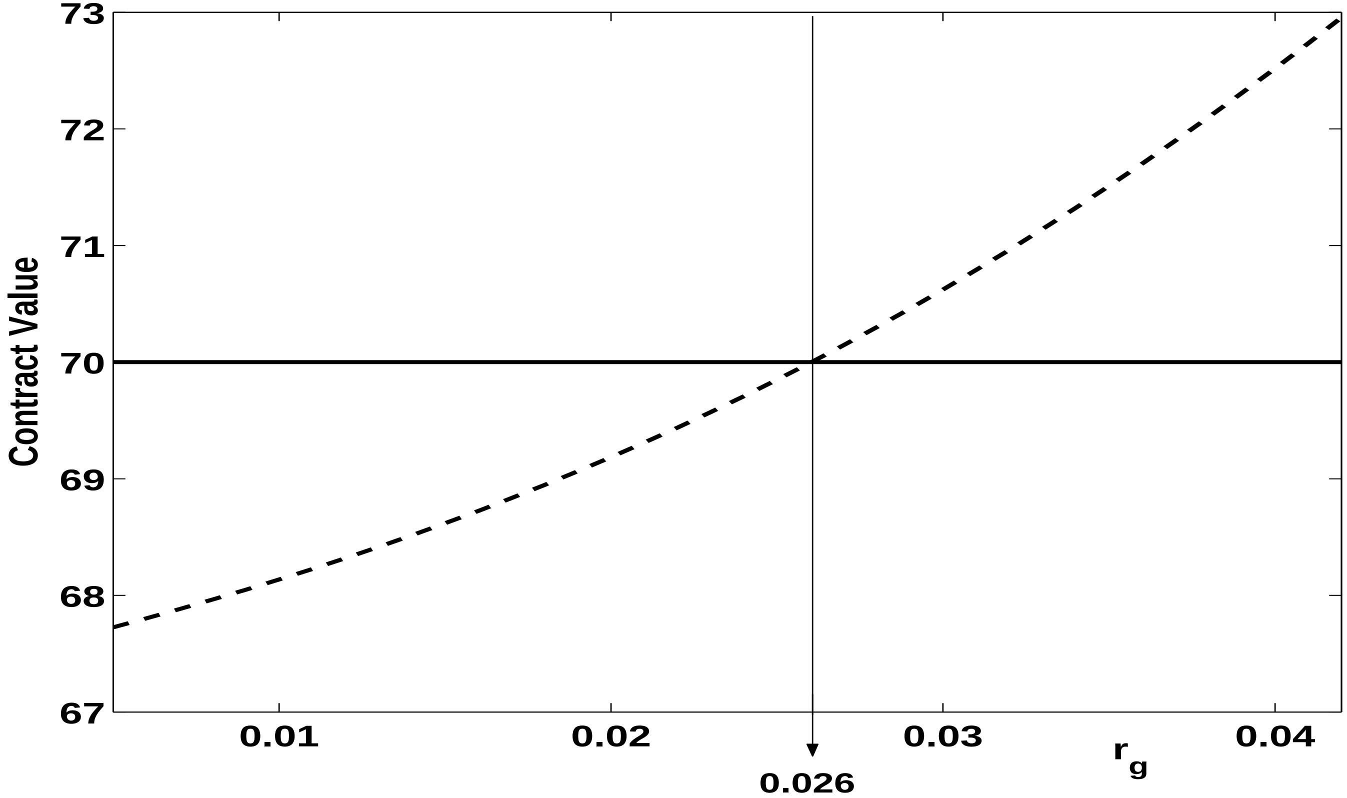
- r_g : minimum guaranteed interest rate
- δ : participating benefits

cannot be fixed arbitrarily : they obey regulatory constraints, and need to be set in such a way as to make the contract fair between the insurer and the policyholder.

Contract Value w.r.t. δ the participating coefficient



Contract Value w.r.t. r_g
Minimum Guaranteed Rate



How to fix the Parameters of a Fair Contract ?

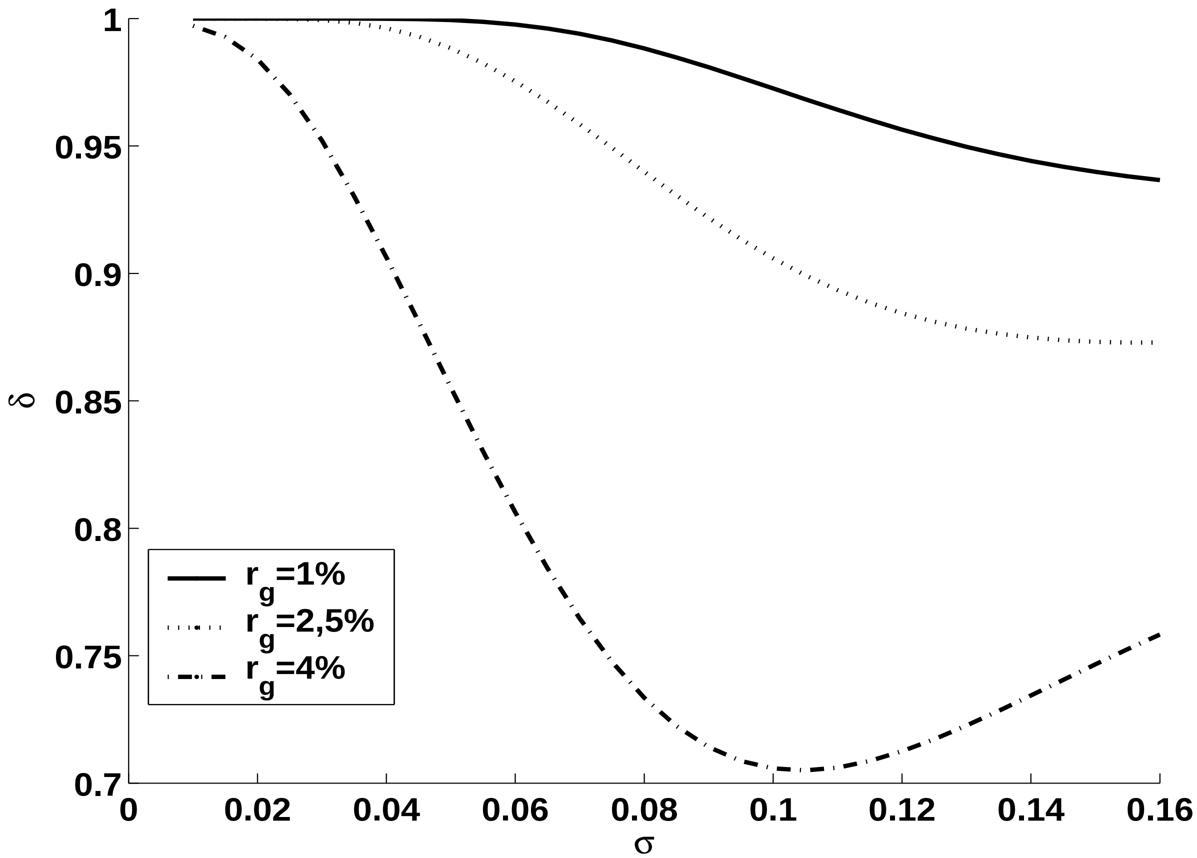
We use a root search algorithm on the following equation to find the fair value of a parameter, *ceteris paribus* :

$$L_0 = \{\text{Contract Value at } t = 0\}$$

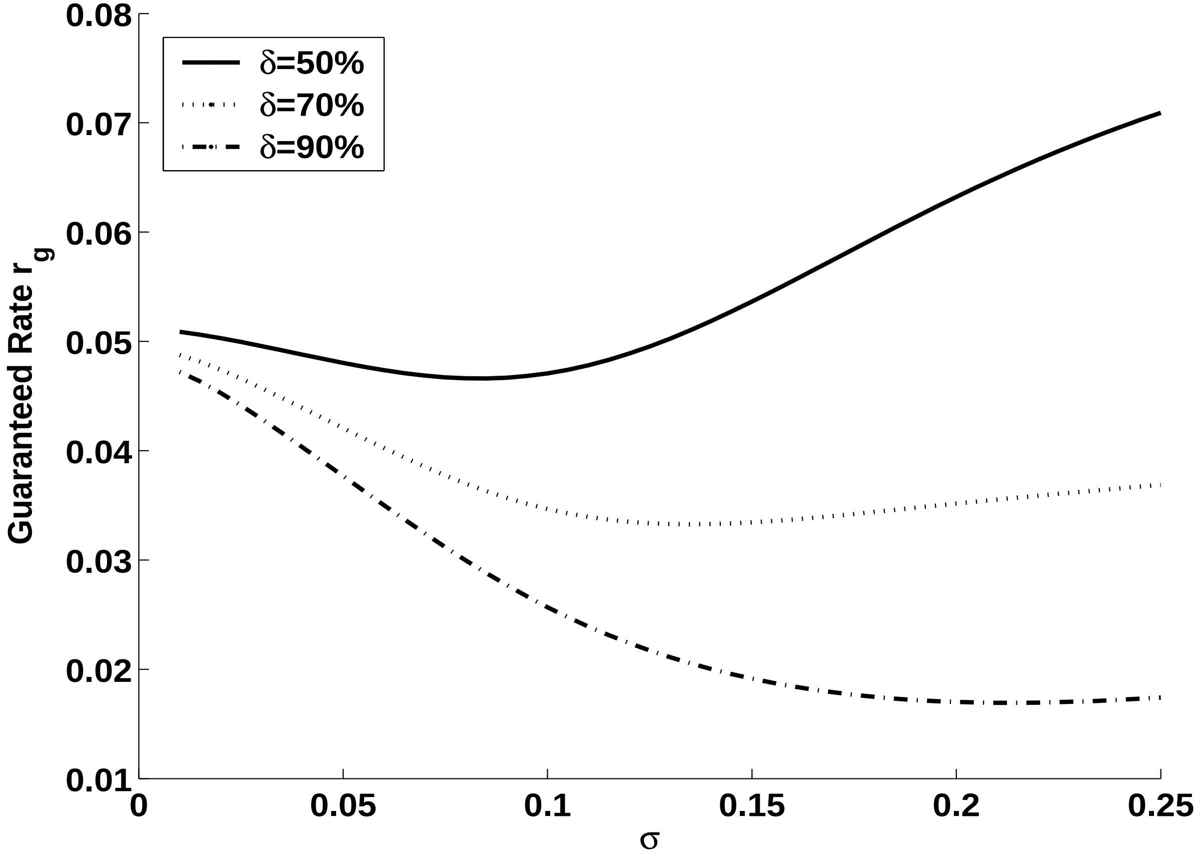
The contract value at $t = 0$ depends on :

- the initial structure of the company : A_0, α, the interest rate parameters and the correlation ρ between the assets and the interest rate,
- the contract maturity T, the barrier level λ,
- some parameters σ, δ and r_g that we will study more in details in the following.

δ with respect to σ



r_g with respect to σ



Conclusion

- A study of relevance in the context of the new IAS and IFRS Standards
- A new method to price standard life insurance guarantees (guaranteed capital and minimum rate with participating bonuses when interest rates are stochastic and the possible default of the company is taken into account)
- The next step is to price supplementary options typical to life insurance contracts (surrender and conversion options, capital paid upon death and not at a fixed time making necessary the use of mortality tables and so on)