Development and Pricing of a New Participating Contract

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Outline of the Talk

1. Bibliography

2. Presentation and Pricing of Standard Contracts

3. Building of New Contracts

4. Obtaining Closed-Form Formulae

5. Conclusion and Numerical Analysis

The Actuarial Bibliography

- Brennan and Schwartz [1976]
- Briys and de Varenne [1993, 1997]
- Grosen and Jørgensen [1997, 2000, 2002]
- Tanskanen and Lukkarinen [2003]
- Jørgensen [...], Ballotta [...], Bacinello [...]
- Bernard, Le Courtois and Quittard-Pinon [2005]

The Finance Bibliography

- Merton [1974]
- Heath, Jarrow and Morton [1992]
- Longstaff and Schwartz [1995]
- Collin-Dufresne and Goldstein [2001]
- Jeanblanc, Yor and Chesney [2005]

Standard Contracts (the Issuing Vehicle)

Assets	Liabilities
A_0	$E_0 = (1-\alpha)A_0$ $L_0 = \alpha A_0$

- E_0 = initial equity value
- L_0 = initial policyholders' investment

Standard Contracts (Minimum Rate)

Existence of a minimum guaranteed rate r_g :

$$L_T^g = L_0 \, e^{r_g T} \quad \text{at } T$$

➠ **Solvency at time T : $\mathbf{A_T \geq L_T^g}$**

Policyholders receive L_T^g

➠ **Default at time T : $\mathbf{A_T < L_T^g}$**

Policyholders receive A_T

Standard Contracts (Participating Bonus)

Policyholders are given a part δ of the benefits when

$$A_T > \frac{L_T^g}{\alpha} \quad \text{where} \quad \alpha < 1.$$

Policyholders now receive at T :

$$\Theta_L(T) = \begin{cases} A_T & \text{if } A_T < L_T^g \\ L_T^g & \text{if } L_T^g \leq A_T \leq \frac{L_T^g}{\alpha} \\ L_T^g + \delta(\alpha A_T - L_T^g) & \text{if } A_T > \frac{L_T^g}{\alpha} \end{cases}$$

Standard Contracts (Early Default)

The firm pursues its activities until T if :

$$\forall t \in [0, T[\quad , \quad A_t > L_0 e^{r_g t} \triangleq B_t$$

Let τ be the default time

$$\tau = \inf\{t \in [0, T] \ / \ A_t < B_t\}$$

In case of prior insolvency, policyholders receive :

$$\Theta_L(\tau) = \ L_0 e^{r_g \tau}$$

Standard Contracts (Valuation Formula)

- With a Continuously Compounded Minimum Rate,
- a Participation in the Assets Performance,
- and the Possibility of an Early Default, we have :

$$V_L(0) = \mathbb{E}_Q \Big(e^{-\int_0^T r_s ds} \left[L_T^g + \delta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+\right] \mathbb{1}_{\tau \geq T} + e^{-\int_0^\tau r_s ds} \left[L_0\, e^{r_g \tau}\right] \mathbb{1}_{\tau < T} \Big)$$

Standard Contracts
(Pricing Methodology)

To price this contract, one has to compute expectations like :

$$\mathbb{E}_{\mathbb{Q}_T}\left[A_T \ \mathbb{1}_{A_T < L_T^g} \ \mathbb{1}_{\tau < T}\right]$$

- ➠ this is a 2D problem in (r, τ)
- ➠ when A is lognormal and correlated to an HJM r ;
- ➠ an extension of Collin-Dufresne and Goldstein [2001]
- ➠ solves the problem in terms of a recurrence equation,
- ➠ Yet, No Closed-Form Formulae Can Be Obtained

Introducing...

... A New Contract

which is :

- only very slightly different from the previous one
- in a totally identical framework for A and r
- where only the Guaranteed Amount is modified
- and Indexed on Government Zero-Coupon Bonds

The New Contract's Guaranteed Amount

The Guarantee is the one of an equivalent position in $\frac{\beta L_0}{P(0,T)}$ Government ZC Bonds Maturing at time T

This Guarantee is worth $l_0^g = \beta L_0$ at time 0,

It is worth $l_t^g = \frac{\beta L_0}{P(0,T)} P(t,T)$ at time t,

And it is worth $l_T^g = \frac{\beta L_0}{P(0,T)}$ at time T

The New Contract's Guaranteed Amount

What is the main Implication of Introducing a Guarantee such as the one Defined in the Previous Slide ?

Indeed the default time becomes :

$$\tau = \inf \{ \; s < T \; / \; A_s < \lambda_1 \, l_s^g \; \}$$

and this allows pricing in closed-form :

$$V'(0) \; = \; \mathbb{E}_{\mathbb{Q}} \Big[e^{-\int_0^T r_s ds} \left(l_T^g + \delta \left(\alpha A_T - l_T^g \right)^+ - \left(l_T^g - A_T \right)^+ \right) \mathbb{1}_{\tau \geq T} + \; e^{-\int_0^\tau r_s ds} \; \lambda_1 \, \lambda_2 \, l_\tau^g \; \mathbb{1}_{\tau < T} \Big] \; !!!$$

Reminder
Underlying Model

The dynamics under Q of the ZC bonds $P(t,T)$ are :

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \sigma_P(t,T) dZ_1^{\mathsf{Q}}(t)$$

whilst the assets dynamics under Q are :

$$\frac{dA_t}{A_t} = r_t dt + \sigma dZ^{\mathsf{Q}}(t)$$

where Z^{Q} and Z_1^{Q} are correlated Q-Brownian motions. $(dZ^{\mathsf{Q}}.dZ_1^{\mathsf{Q}} = \rho dt)$.

Reminder
Underlying Model

Now, under the T-forward-neutral measure $\mathbb{Q}_T$,
and after decorrelating the driving Brownian motions :

$$\frac{dP(t,T)}{P(t,T)} = (r_t + \sigma_P^2(t,T))dt - \sigma_P(t,T)dZ_1^{\mathbb{Q}_T}$$

and

$$\frac{dA_t}{A_t} \quad = (r_t - \sigma\rho\sigma_P(t,T))dt + \sigma\left(\rho dZ_1^{\mathbb{Q}_T} + \sqrt{1-\rho^2}dZ_2^{\mathbb{Q}_T}\right)$$

The New Contract
Back to the Valuation

So, how can V' be priced in closed-form ?

We Illustrate our Approach by Computing the Forward-Neutral Ruin Probability

$$\mathrm{Q}_T \left(\inf_{u \in [0,T[} \left(\frac{A_u}{P(u,T)} \right) < \lambda_1 \, l_T^g \right)$$

The New Contract
Back to the Valuation

The solution of our problem lies in the fact that :

$$\frac{A_u}{P(u,T)} = \frac{A_0}{P(0,T)} e^{N_u - \frac{1}{2}\xi(u)}$$

where the differential of the martingale N is defined by :

$$dN_s = (\sigma_P(s,T) + \rho\sigma)\, dZ_1^{\mathbb{Q}_T}(s) + \sigma\sqrt{1-\rho^2}\, dZ_2^{\mathbb{Q}_T}(s)$$

and the quadratic variation of N is :

$$\xi(u) \; = < N >_u = \int_0^u [(\sigma_P(s,T) + \rho\sigma)^2 + \sigma^2(1-\rho^2)]ds$$

The New Contract
Back to the Valuation

$$
\begin{aligned}
FNRP &= \mathrm{Q}_T\left(\inf_{u\in[0,T[}\left(\frac{A_u}{P(u,T)}\right) < \lambda_1\, l_T^g\right)\\
&= \mathrm{Q}_T\left\{\min_{u\in[0,T]}\left(\frac{A_0}{P(0,T)}e^{N_u-\frac{1}{2}\xi(u)}\right) < \lambda_1\, l_T^g\right\}\\
&= \mathrm{Q}_T\left\{\min_{u\in[0,T]}\left(e^{B_{\xi(u)}-\frac{1}{2}\xi(u)}\right) < \frac{P(0,T)\,\lambda_1\, l_T^g}{A_0}\right\}\\
&= \mathrm{Q}_T\left\{\min_{s\in[0,\xi(T)]}\left(B_s-\frac{1}{2}s\right) < \ln\left(\lambda_1\,\beta\alpha\right)\right\}
\end{aligned}
$$

where Dubins-Schwarz is the Key Theorem

Conclusion
On the Contract Valuation

The contract can be priced as a linear combination of Gaussian functions

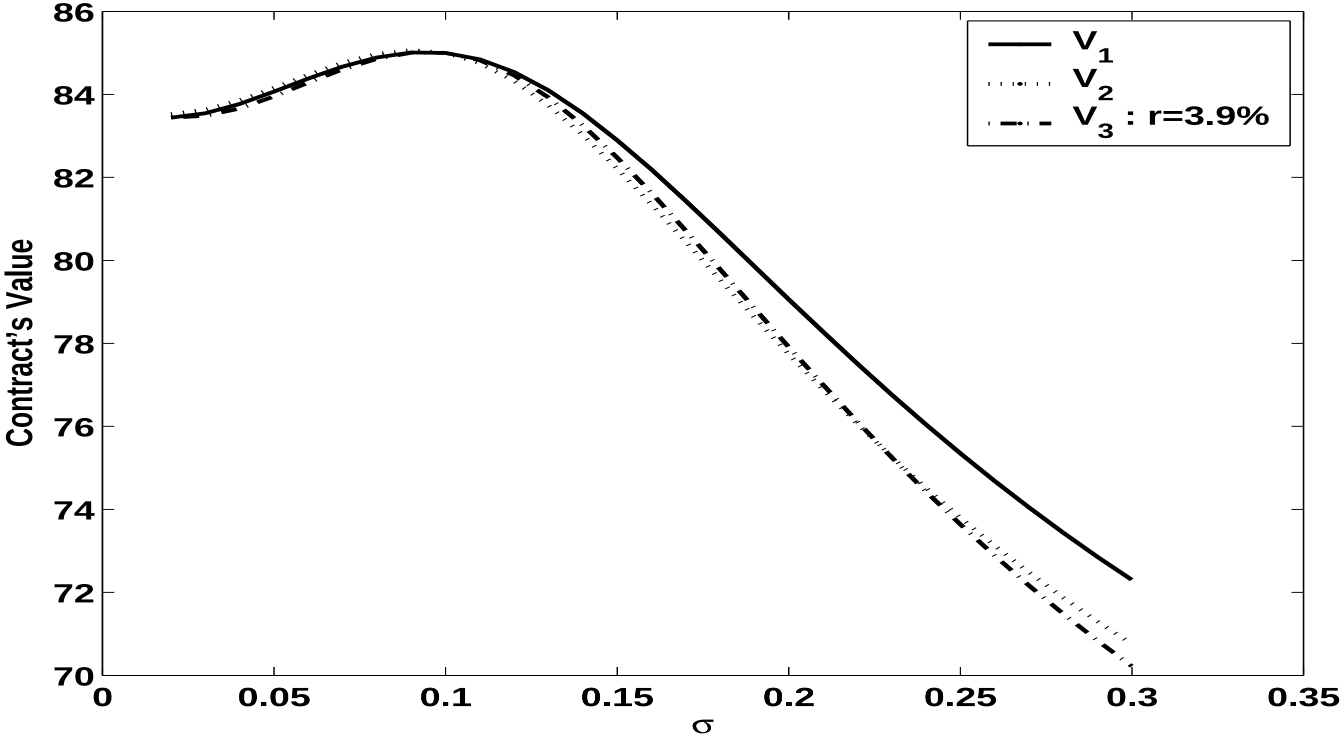
The pricing is therefore instantaneous

We presume this could be useful in Fund Valuation and Modeling under Cushion Insurance

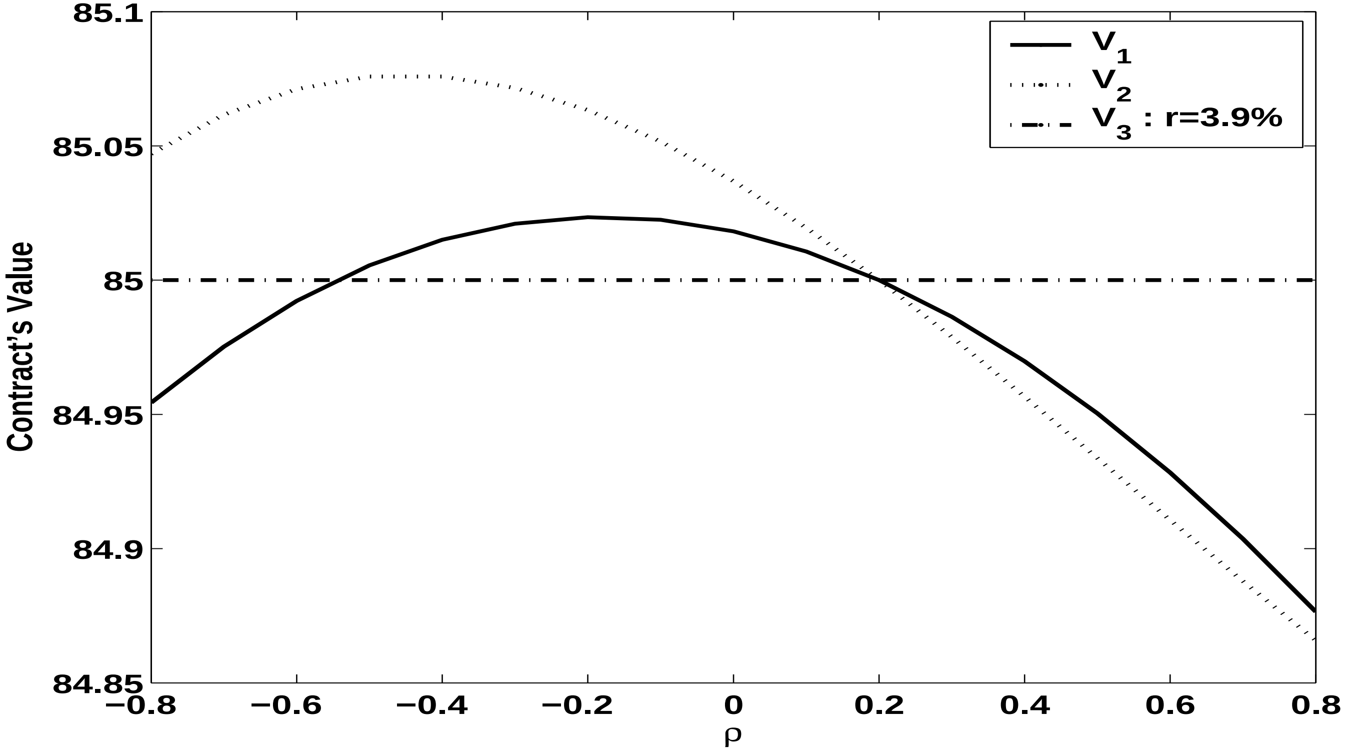
Yet, already notice this covenant modification improves a Participating Contract Characteristics...

... because our barrier moves with the interest rates ...

Contract Value w.r.t. σ (the Assets volatility)



Contract Value w.r.t. ρ (Correlation A / r)



General Conclusion

A Proposal for New Participating Contracts

Instead of guaranteeing at time t an amount proportional to $e^{r_g t}$, guarantee an amount proportional to $P(t, T)$

In such a setting, closed-form formulae can be obtained

This method also allows to price exotic options (sharks options for instance)